A Tutorial on Mixed-Integer Linear Optimization

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Research Center MATHEON

Mathematics for key technologies

Group Seminar, 21 June 2016

- Mathematical optimization/programming problem

$$\max\{g(x) \mid f_j(x) \leq 0, x \in D\} \text{ or } \min\{g(x) \mid f_j(x) \leq 0, x \in D\}$$

with $D \subseteq \mathbb{R}^n$, $g, f_j : D \to \mathbb{R}$, $j = 1, \ldots, m$.

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Optimization Problems

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- Feasible/optimal solutions
  - need not exist,
  - need not be unique.

Linear optimization

$$
\begin{array}{llllllll}
\max & c_1 x_1 & + & \cdots & + & c_n x_n & & \\
\text{w.r.t.} & a_{11} x_1 & + & \cdots & + & a_{1n} x_n & \leq & b_1, \\
& \vdots & & & & \vdots & & \\
& a_{m1} x_1 & + & \cdots & + & a_{mn} x_n & \leq & b_m, \\
& x_1, & & \ldots & & , x_n & \in & \mathbb{R}.
\end{array}
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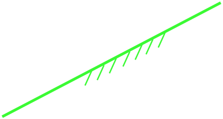
In matrix notation:

$$\max\{c^T x \mid Ax \leq b, x \in \mathbb{R}^n\},$$

with $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$.

Geometric illustration



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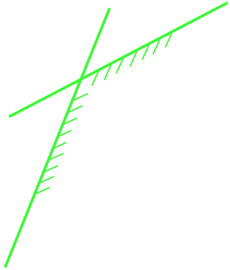

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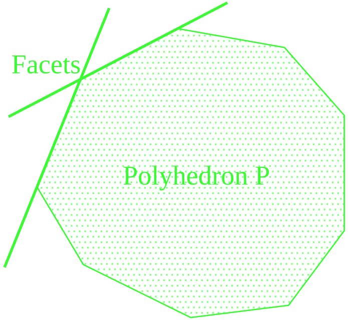

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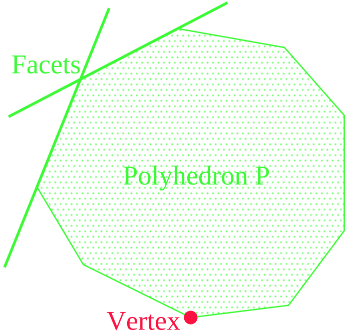

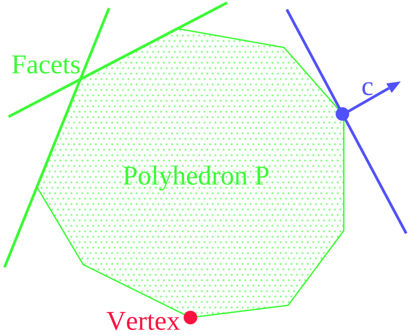

- Polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ $\rightsquigarrow$ feasible region
- Linear optimization problem (LP): $\max\{c^T x \mid Ax \leq b, x \in \mathbb{R}^n\}$

Linear optimization problem

$$\max\{c^T x \mid Ax \leq b, x \in \mathbb{R}^n\} \tag{LP}$$

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Simplex-Algorithm (Dantzig 1947)

1. Find a vertex of the polyhedron P.
2. Proceed from vertex to vertex along edges of P such that the objective function $g = c^T x$ increases.
3. Either a vertex will be reached that is optimal, or an edge will be chosen which goes off to infinity and along which g is unbounded.

Complexity of linear optimization

- All known variants of the Simplex algorithm have worst-case exponential running time.

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- In practice, the Simplex algorithm is very efficient.
- There exist polynomial-time algorithms for linear optimization
  - Ellipsoid method (Khachiyan 1979)
  - Interior point methods (Karmarkar 1984)

- Linear optimization/programming (LP) polynomial

$$\max\{c^T x \mid Ax \leq b, x \in \mathbb{R}^n\}$$

Integer linear optimization

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$$\max\{c^T x \mid Ax \leq b, x \in \mathbb{R}^n\}$$

- Integer linear optimization (IP) NP-hard

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Integer linear optimization

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$$\max\{c^T x \mid Ax \leq b, x \in \mathbb{R}^p \times \mathbb{Z}^q\}$$

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- Mixed 0-1 linear optimization — NP-hard

$$\max\{c^T x \mid Ax \le b, x \in \mathbb{R}^p \times \{0,1\}^q\}$$

Integer hull

- $P = \{x \in \mathbb{R}^n \mid Ax \leq b\},\ A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m$ polyhedron
- Integer points in $P \quad \rightsquigarrow \quad P \cap \mathbb{Z}^n$

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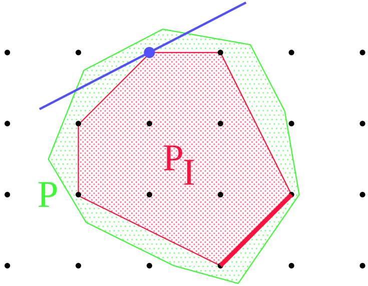

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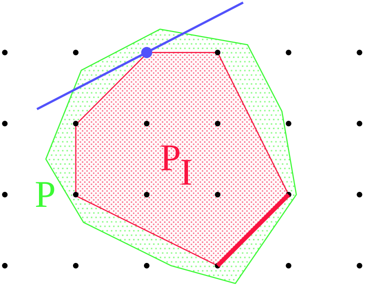

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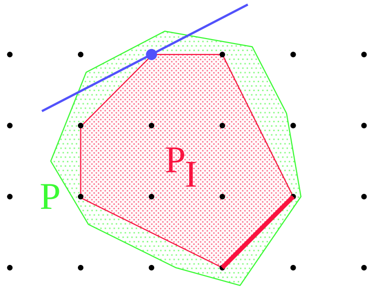

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- P_I is a polyhedron: $P_I = \{x \in \mathbb{R}^n \mid \tilde{A}x \leq \tilde{b}\}, \tilde{A} \in \mathbb{Z}^{k \times n}, \tilde{b} \in \mathbb{Z}^k$

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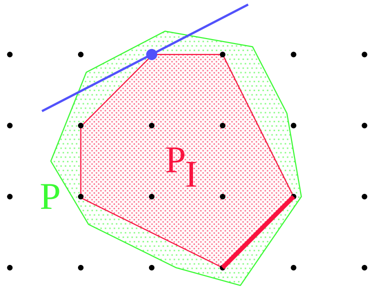

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$$\max\{c^T x \mid Ax \leq b, x \in \mathbb{Z}^n\} = \max\{c^T x \mid \tilde{A}x \leq \tilde{b}, x \in \mathbb{R}^n\}$$

P_I is very hard to compute $\rightsquigarrow$ approximation by cutting planes (Gomory 1958)

Cutting Planes

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- ▷ Solve the linear relaxation $\max\{c^T x \mid Ax \leq b, x \in \mathbb{R}^n\}$ and find an optimal solution x^*.

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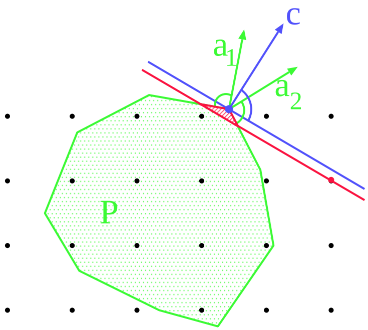
- If $x^* \in \mathbb{Z}^n$, the integer linear program has been solved.
- If $x^* \notin \mathbb{Z}^n$, generate a cutting plane $a^T x \leq \beta$, which is satisfied by all integer points in P, but which cuts off the non-integer vertex x^*.



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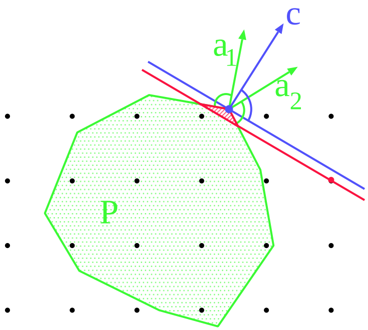
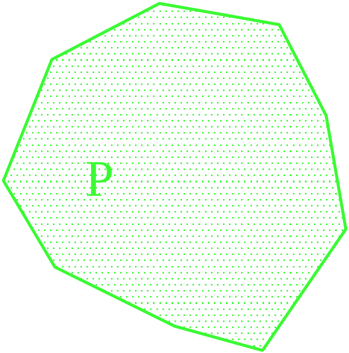
- Add the inequality $a^T x \leq \beta$ to the system $Ax \leq b$ and solve the relaxation again.



Gomory 58, Chvátal 73



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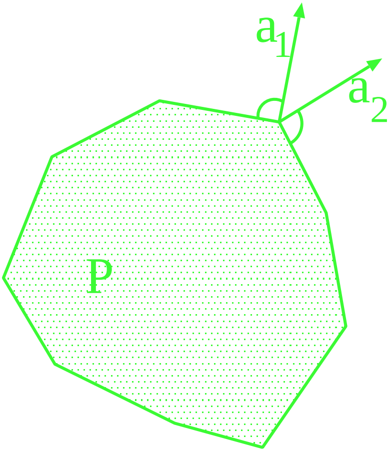

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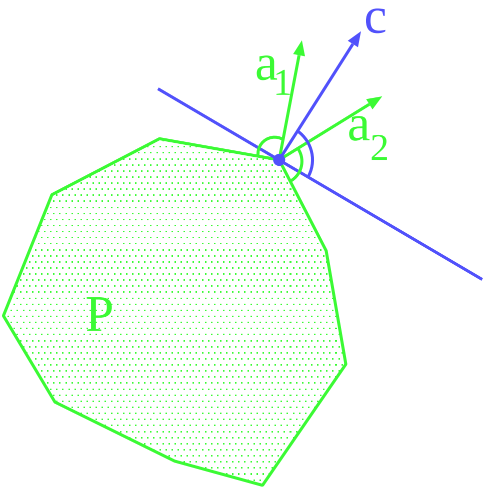

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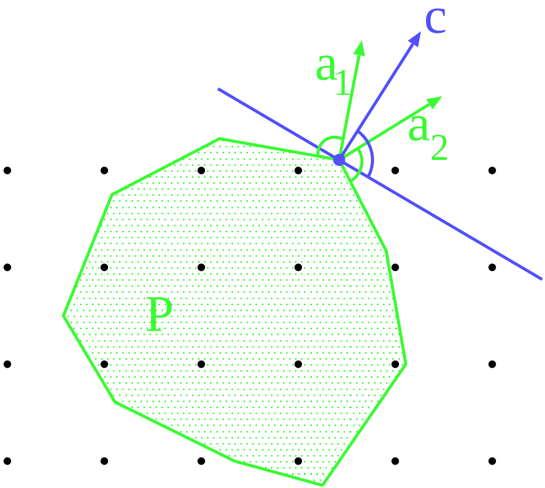

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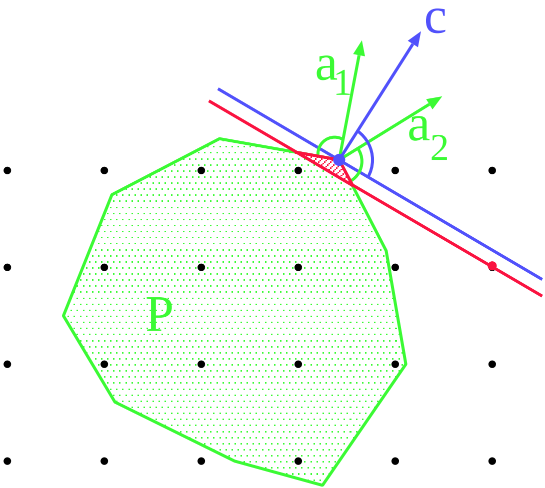

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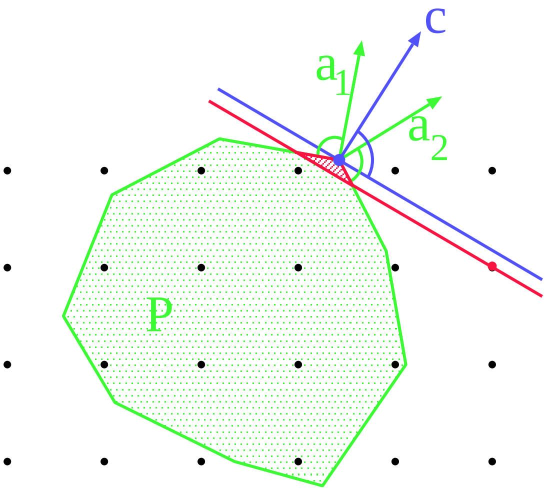

$$\frac{Ax \leq b}{u^T A x \leq \lfloor u^T b \rfloor}$$

$$\text{if } \begin{cases} u \geq 0 \\ u^T A \in \mathbb{Z}^n \end{cases}$$

Branch-and-Bound

Land/Doig 1960

- Divide the set of feasible solutions into subsets ("branch")
- Compute bounds for the objective function on these subsets ("bound") $\rightsquigarrow$ linear relaxation !
- Use these bounds to discard some subsets from further consideration.

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Basic principle

- $S = S^0 \cup S^1$
- Local upper bound: $\max\{c^T x \mid x \in S^0\} \leq UB^0$
- Global lower bound (feasible solution): $x^* \in S$, with $c^T x^* = LB > UB^0$

$\rightsquigarrow$ $x_{opt} \in S^1$.

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Grötschel, Padberg, Rinaldi, . . . , 1980's

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- Software: CPLEX, Gurobi, SCIP, . . .

- Linear optimization polynomial
  - Polyhedra
  - Simplex algorithm, interior point methods

Summary

- Linear optimization — polynomial
  - Polyhedra
  - Simplex algorithm, interior point methods
- (Mixed-)integer linear optimization — NP-hard
  - Integer hull and linear relaxation
  - Cutting planes
  - Branch-and-bound
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- Linear optimization — polynomial
  - Polyhedra
  - Simplex algorithm, interior point methods
- (Mixed-)integer linear optimization — NP-hard
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  - Branch-and-bound
  - Branch-and-cut
- Many applications in bioinformatics and systems biology
  - Flux balance analysis
  - Elementary flux modes
  - ……