III. Matching

- $G = (V, E)$ undirected graph
- Matching: Subset of edges $M \subseteq E$, no two of which share an endpoint.
- Maximum (cardinality) matching: Matching of maximum cardinality
- Perfect matching: Every vertex in $V$ is matched.
- Maximum weighted matching: Given a weight function $w : E \rightarrow \mathbb{R}$, find a matching $M$ such that $w(M) = \sum_{e \in M} w(e)$ is maximal.

Augmenting paths

- Let $M$ be a matching in $G = (V, E)$.
- A path $P = (v_0, v_1, ..., v_t)$ in $G$ is called $M$-augmenting if:
  - $t$ is odd,
  - $v_1 v_2, v_3 v_4, ..., v_{t-2} v_{t-1} \in M$,
  - $v_0, v_t \not\in \bigcup_{e \in M} e$.
- If $P$ is an $M$-augmenting path and $E(P)$ the edge set of $P$, then
  $M' = M \triangle E(P) = (M \setminus E(P)) \cup (E(P) \setminus M)$
  is a matching in $G$ of size $|M'| = |M| + 1$.

Berge’s Theorem

Theorem (Berge 1957)
Let $M$ be a matching in the graph $G = (V, E)$. Then either $M$ is a maximum cardinality matching or there exists an $M$-augmenting path.

Generic Matching Algorithm

Initialization: $M \leftarrow \emptyset$
Iteration: If there exists an $M$-augmenting path $P$, replace $M \leftarrow M \triangle E(P)$.

~ how can one find an $M$-augmenting path?

- Difficult in general ~ Edmonds’ matching algorithm (Edmonds 1965)
- Easy for bipartite graphs

Bipartite graphs

A graph $G = (V, E)$ is bipartite if there exist $A, B \subseteq V$ with $A \cup B = V, A \cap B = \emptyset$ and each edge in $E$ has one end in $A$ and one end in $B$.

Proposition
A graph $G = (V, E)$ is bipartite if and only if each circuit of $G$ has even length.
Bipartite matching

Matching augmenting algorithm for bipartite graphs

Input: Bipartite graph $G = (A \cup B, E)$ with matching $M$.
Output: Matching $M'$ with $|M'| > |M|$ or proof that no such matching exists.

Description: Construct a directed graph $D_M$ with the same node set as $G$.
For each edge $e = \{a, b\}$ in $G$ with $a \in A, b \in B$:
- if $e \in M$, there is the arc $(b, a)$ in $D_M$.
- if $e \not\in M$, there is the arc $(a, b)$ in $D_M$.
Let $A_M = A \setminus \bigcup M$ and $B_M = B \setminus \bigcup M$.
$M$-augmenting paths in $G$ correspond to directed paths in $D_M$ starting in $A_M$ and ending in $B_M$.

Theorem
A maximum-cardinality matching in a bipartite graph $G = (V, E)$ can be found in time $O(|V||E|)$.

Bipartite matching as a maximum flow problem

- Add a source $s$ and edges $(s, a)$ for $a \in A$, with capacity 1.
- Add a sink $t$ and edges $(b, t)$ for $b \in B$, with capacity 1.
- Direct edges in $G$ from $A$ to $B$, with capacity 1.

- Integral flows $f$ correspond to matchings $M$, with $\text{val}(f) = |M|$.
- Ford-Fulkerson takes time $O(|V||E|)$, since $v^* \leq |V|/2$.
- Can be improved to $O(\sqrt{|V||E|})$ (Hopcroft-Karp 1973).

Marriage theorem

Theorem (Hall 1935)
A bipartite graph $G = (A \cup B, E)$, with $|A| = |B| = n$, has a perfect matching if and only if for all $B' \subseteq B$, $|B'| \leq |N(B')|$, where $N(B')$ is the set of all neighbors of nodes in $B'$.
Proof

- Let \((S, T)\) be an \((s, t)\)-cut in the corresponding network.
- Define \(A_S = A \cap S, A_T = A \cap T, B_S = B \cap S, B_T = B \cap T\).
- Show \(\text{cap}(S, T) \geq n\) (Exercise)
- By the max-flow min-cut theorem, the maximum flow is at least \(n\).

\[
\begin{align*}
A_S &= A \cap S \\
A_T &= A \cap T \\
B_S &= B \cap S \\
B_T &= B \cap T
\end{align*}
\]

Network connectivity: Menger’s theorems

- \(G = (V, E)\) directed graph, \(s, t \in V, s \neq t\) non-adjacent.
- **Theorem** (Menger 1927) The maximum number of *arc-disjoint* paths from \(s\) to \(t\) equals the minimum number of arcs whose removal disconnects all paths from \(s\) to \(t\).
- **Theorem** (Menger 1927) The maximum number of *node-disjoint* paths from \(s\) to \(t\) equals the minimum number of nodes (different from \(s\) and \(t\)) whose removal disconnects all paths from \(s\) to \(t\).

References and further reading