A sharp interface finite volume method for
variable density zero Mach number two-phase flow
with surfactant-dependent surface tension

Rupert Klein, Ralf Kornhuber
Anna Hartkopf, Matthias Waidmann, Stephan Gerber
Freie Universität Berlin, Institut für Mathematik

Sharp interface finite volume method for isothermal two-phase flow

1a) Nonlocal Navier-Stokes pressure projection boundary conditions

Cooperation: M. Abls, Regensburg

Viscous, constant density ($\rho = 1$) flow in a closed domain $\Omega$:

\[ \nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega \]

\[ \mathbf{v} = \mathbf{0} \quad \text{on } \partial \Omega \]

Pressure field satisfies, see Li et al. (2007)

\[ \nabla p = \nabla \cdot (\rho \mathbf{v}) \quad \text{in } \Omega \]

\[ \nabla p = \nabla \cdot (\rho \mathbf{v}) \quad \text{on } \partial \Omega \]

not merely $\nabla \cdot \mathbf{v} = 0$ on $\partial \Omega$. This implies artificial pressure/velocity boundary layers in splitting or projection methods. Improved projection (Brown et al. (2001)) and “consistent splitting” schemes (e.g., Liu et al. (2010)) reduce the inconsistency to higher-order errors. Only fully implicit schemes currently achieve full force balance at a no-slip wall.

Implicit pressure boundary condition in a projection method:

With initial data $\mathbf{v}^{(0)}$, the predictor step solves, over one time step,

\[ \mathbf{v}^{(n+1)} = \mathbf{v}^{(n)} + \Delta t \nabla p^{(n)} \quad \text{in } \Omega \]

\[ \mathbf{v}^{(n+1)} = \mathbf{0} \quad \text{on } \partial \Omega \]

The resulting velocity $\mathbf{v}$ is corrected by a pressure gradient force, so that

\[ \mathbf{v}^{(n+1)} = \mathbf{v}^{(n)} + \Delta t \nabla p^{(n+1)} \quad \text{with } \nabla \cdot \mathbf{v}^{(n+1)} = 0 \quad \text{in } \Omega \]

\[ \mathbf{v}^{(n+1)} = \mathbf{0} \quad \text{on } \partial \Omega \]

Imposing a third-order boundary condition for a second-order equation should be ill-posed? Yet for a rigid, straight wall we find a Helmholtz equation for $\nabla \cdot \mathbf{v}$ on the boundary

\[ \nabla \cdot \mathbf{v}^{(n+1)} = \nabla \cdot \mathbf{v}^{(n)} \]

How does this affect accuracy near boundaries, and does it carry over to more complex situations?

1b) Asymptotic preconditioner for large jumps of fluid properties across an interface

Cooperation Jinchao Xu, Pennsylvania State & Peking University

Loading-order problem

\[ \bar{p}_0 = p_0 \quad \text{on } \partial \Omega \]

\[ \nabla \cdot (\rho \mathbf{v}) \quad \text{in } \Omega \]

\[ \nabla \cdot (\rho \mathbf{v}) \quad \text{on } \partial \Omega \]

First-order problem

\[ \nabla \cdot (\rho \mathbf{v}) = f \quad \text{in } (\Omega \cup \Sigma) \]

\[ \nabla \cdot (\rho \mathbf{v}) = 0 \quad \text{on } \Sigma \]

\[ p = \bar{p}_0 + \bar{q} \quad \text{on } \Sigma \]

First step:

Comparison with domain decomposition approaches (structurally and quantitatively)

Efficiency: Simultaneous iterative solution with exact interface condition

\[ \frac{\partial \mathbf{v}}{\partial n} = \mathbf{v} \quad \text{on } \Sigma, \quad p = 0, 1, \ldots \]

The natural AMR grid hierarchy suggests a multilevel version:

• Replace coupled solution by coupled smoothing on each refinement level

• Convergence analysis: Reformulation in terms of multilevel preconditioning

Conservative discretization of the surfactant transport equation

2a) Conservative characteristic DG for advection on a moving surface

Coop: M. Oberlack, Y. Wang, Darmstadt, M. Oevermann, Chalmers

Return to original idea (1st. proposes) of representing surfactant distribution directly on the surface.

Issues:

• Surfactant conservation

• Evolving cell/interface cuts

Borrowing ideas from van Leer (1977), Prather (1984) we plan to address these issues as follows:

• Polynunomial

• Piecewise polynomial representation of cell surfaces after advection

are LS-projected back to the polynomial ansatz space.

2b) Conservative DG method for diffusion on a moving surface

Diffusive transport via operator splitting

\[ C_t + \nabla \cdot (\mathbf{v} \nabla C) = f \quad \text{in } \Omega \]

\[ C_t + \nabla \cdot (\mathbf{v} \nabla C) = f \quad \text{on } \partial \Omega \]

Time discretization: Rosenbrock method (2. order, L-stable)

\[ (k_1) \mathbf{v}(k_1) = \frac{1}{\Delta t} \left( C_t(k_1) \right) \quad \text{in } \Omega \]

\[ (k_1) \mathbf{v}(k_1) = \frac{1}{\Delta t} \left( C_t(k_1) \right) \quad \text{on } \partial \Omega \]

DG method

\[ a_{DG}(v, w) = \sum_{r} A_r \left( \nabla v, \nabla w \right) \quad \text{in } \tilde{E} \]

with broken finite element space

\[ v, w \in S_h \quad \text{for } v \in \mathcal{V} \quad \text{and } w \in \mathcal{V} \quad \text{on } \tilde{T} \]

Practical assembly of stiffness matrix

Algebraic solution:

- direct method, e.g., PARDISO

Perspective: Diffusive transport via characteristic method

2c) Soluble surfactant transport

Transport equation in the bulk:

\[ C_t + \nabla \cdot (\mathbf{v} C) = f \quad \text{in } \Omega \]

Transport equation on the interface:

\[ C_t + \nabla \cdot (\mathbf{v} C) = f \quad \text{on } \partial \Omega \]

Coupling conditions:

\[ \left. \mathbf{v} \cdot n \right|_{\partial \Omega} = 0 \quad \text{(Henry condition)} \]

\[ \left. \bar{q} \right|_{\partial \Omega} = \frac{g}{\sigma} \quad \text{(adsorption/desorption rate)} \]

Tools developed in this project for mass conserving discretization

• Elliptic/parabolic solver for bulk diffusion – same technology as used for the pressure projections with local enforcement of jump conditions.

• Mass-preserving, DG-in-surface diffusive transport will be extended to account for source terms (from diffusion into the surface / adsorption).

• Challenge: formulation of locally conservative discrete coupling

References


M.J. Gander, C. Japhet: An Algorithm for Non-Matching Grid Projections with


M.J. Gander, C. Japhet: A Multigrid Algorithm for Solving Elliptic Problems with


