You have 90 minutes for the exam. Please write Matrikelnummer and name on each sheet you hand in.
A plant makes aluminum and copper wire. Each kg of aluminum wire requires 5 kWh of electricity and 1/4 hr. of labor. Each kg of copper wire requires 2 kWh of electricity and 1/2 hr. of labor. Production of copper wire is restricted by the fact that raw materials are available to produce at most 60 kg/day. Electricity is limited to 500 kWh/day and labor to 40 hrs./day. If the profit from aluminum wire is $0.25/kg, and the profit from copper is $0.40/kg., how much of each should be produced to maximize profit and what is the maximum profit?

(a) • Model the problem as a linear program.
   • Solve the linear program graphically to compute the coordinates of the optimal solution as well as its value.

(b) • Formulate the dual of your LP.
   • State the strong duality theorem and weak duality theorem in linear programming.
Given a graph $G = (V, E)$, two vertices $s, t \in V$, and a set of pairs of vertices $C \subset V \times V$, the shortest antisymmetric path problem consists in finding a path from $s$ to $t$ with the minimal number of edges, which contains at most one vertex from each pair of vertices in $C$.

(a) Give an integer linear programming formulation for the antisymmetric shortest path problem.

(b) How can you, in general prove, whether an inequality for a combinatorial optimization problem is facet-defining?
3. [15 Points] (Combinatorial Optimization: Branch-and-cut)

Solve the *shortest antisymmetric path problem* for the small instance given below by branch-and-cut where you use the forbidden pairs inequalities as cutting planes. The forbidden pairs of vertices are denoted by dashed lines.
4. [6+6=12 Points] (Combinatorial Optimization: Lagrange relaxation)

Assume you are given the optimization problem

\[
\begin{align*}
\text{min } & \quad c^T x \\
\text{subject to } & \quad Ax \geq b \\
& \quad Dx \geq d \\
& \quad x \text{ integer}
\end{align*}
\]

with \( A, D, b, c, d \) having integer entries. Let \( Z_{IP} \) be the optimal value to the ILP above and let

\[ X = \{ x \text{ integral } | \ Dx \geq d \}. \]

We assume that optimizing over the set \( X \) can be done very easily, whereas adding the bad constraints \( Ax \geq b \) makes the problem intractable.

(a) Formulate the Lagrangian Dual of the above problem.

(b) Show that the solution of the Lagrangian dual, \( Z_D \), is a lower bound for \( Z_{IP} \).
5. [15 Points] (Constraint programming)

Consider the constraint satisfaction problem

\[
x_1 \in \{0, 1, 2\}, \ x_2 \in \{1, 2, 3\}, \ x_3 \in \{0, 1, 2, 3\}
\]
\[
C_1 : x_1 \geq 1, \ C_2 : x_2 < 3,
\]
\[
C_{1,3} : x_1 = x_3, \ C_{2,3} : x_2 < x_3.
\]  

(CSP1)

(a) Draw the corresponding constraint graph.
(b) Make the graph node consistent.
(c) Consider the arcs one by one and make the graph arc consistent.

Consider now the problem

\[
x_1 \in \{1, 2\}, \ x_2 \in \{1, 2\}, \ x_3 \in \{1, 2, 3\},
\]
\[
C_{1,2} : x_1 \neq x_2, \ C_{1,3} : x_1 = x_3, \ C_{2,3} : x_2 \leq x_3.
\]  

(CSP2)

(d) Is the corresponding constraint graph arc consistent? Justify your answer.
(e) Apply the forward checking algorithm to find all solutions.
6. [10 Points] (Metaheuristics)

(a) What is the difference between complete and approximate algorithms for discrete optimization problems?

(b) Briefly describe two metaheuristics of your choice.
(Supplementary sheet 1)