Linear programming

Optimization Problems

• General optimization problem

\[ \max \{ z(x) \mid f_j(x) \leq 0, x \in D \} \text{ or } \min \{ z(x) \mid f_j(x) \leq 0, x \in D \} \]

where \( D \subseteq \mathbb{R}^n, f_j : D \to \mathbb{R} \), for \( j = 1, \ldots, m, z : D \to \mathbb{R} \).

• Linear optimization problem

\[ \max \{ c^T x \mid Ax \leq b, x \in \mathbb{R}^n \}, \text{ with } c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \]

• Integer optimization problem: \( x \in \mathbb{Z}^n \)

• 0-1 optimization problem: \( x \in \{0,1\}^n \)

Feasible and optimal solutions

• Consider the optimization problem

\[ \max \{ z(x) \mid f_j(x) \leq 0, x \in D, j = 1, \ldots, m \} \]

• A feasible solution is a vector \( x^* \in D \subseteq \mathbb{R}^n \) such that \( f_j(x^*) \leq 0 \), for all \( j = 1, \ldots, m \).

• The set of all feasible solutions is called the feasible region.

• An optimal solution is a feasible solution such that

\[ z(x^*) = \max \{ z(x) \mid f_j(x) \leq 0, x \in D, j = 1, \ldots, m \} \]

• Feasible or optimal solutions

  – need not exist,
  – need not be unique.

Transformations

• \( \min \{ z(x) \mid x \in S \} = \max \{ -z(x) \mid x \in S \} \).

• \( f(x) \geq a \) if and only if \( -f(x) \leq -a \).

• \( f(x) = a \) if and only if \( f(x) \leq a \wedge -f(x) \leq -a \).

Lemma

Any linear programming problem can be brought to the form

\[ \max \{ c^T x \mid Ax \leq b \} \text{ or } \max \{ c^T x \mid Ax = b, x \geq 0 \} \].

Proof: a) \( a^T x \leq \beta \iff a^T x + x' = \beta, x' \geq 0 \) (slack variable)
b) \( x \) free \( \iff x = x^+ - x^-, x^+, x^- \geq 0 \).
Practical problem solving

1. Model building
2. Model solving
3. Model analysis

Example: Production problem

A firm produces \( n \) different goods using \( m \) different raw materials.

- \( b_i \): available amount of the \( i \)-th raw material
- \( a_{ij} \): number of units of the \( i \)-th material needed to produce one unit of the \( j \)-th good
- \( c_j \): revenue for one unit of the \( j \)-th good.

Decide how much of each good to produce in order to maximize the total revenue \( \rightarrow \) decision variables \( x_j \).

Linear programming formulation

\[
\begin{align*}
\text{max} & \quad c_1 x_1 + \cdots + c_n x_n \\
\text{w.r.t.} & \quad a_{11} x_1 + \cdots + a_{1n} x_n \leq b_1, \\
& \quad \vdots \\
& \quad a_{m1} x_1 + \cdots + a_{mn} x_n \leq b_m, \\
& \quad x_1, \ldots, x_n \geq 0.
\end{align*}
\]

In matrix notation:

\[
\max \{ c^T x \mid Ax \leq b, x \geq 0 \},
\]

where \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n, x \in \mathbb{R}^n \).

Geometric illustration

\[
\begin{align*}
\text{max} & \quad x_1 + x_2 \\
\text{w.r.t.} & \quad x_1 + 2x_2 \leq 3 \\
& \quad 2x_1 + x_2 \leq 3 \\
& \quad x_1, x_2 \geq 0
\end{align*}
\]