Complexity of linear programming

**Theorem** (Khachyian 79) The following problems are solvable in polynomial time:

- Given a matrix $A \in \mathbb{Q}^{m \times n}$ and a vector $b \in \mathbb{Q}^m$, decide whether $Ax \leq b$ has a solution $x \in \mathbb{Q}^n$, and if so, find one.

- (Linear programming problem) Given a matrix $A \in \mathbb{Q}^{m \times n}$ and vectors $b \in \mathbb{Q}^m$, $c \in \mathbb{Q}^n$, decide whether $\max \{ c^T x \mid Ax \leq b, x \in \mathbb{Q}^n \}$ is infeasible, finite, or unbounded. If it is finite, find an optimal solution. If it is unbounded, find a feasible solution $x_0$, and find a vector $d \in \mathbb{Q}^n$ with $Ad \leq 0$ and $c^T d > 0$.

### Complexity of constraint solving: Overview

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### Integer Linear Optimization (ILP)

- $z_P = \max \{ c^T x \mid Ax \leq b, x \in \mathbb{Z}^n \}, A \in \mathbb{Q}^{m \times n}, b \in \mathbb{Q}^m$

- $z_{LP} = \max \{ c^T x \mid Ax \leq b, x \in \mathbb{R}^n \}$ \hspace{1cm} linear (programming) relaxation

- $P = \{ x \in \mathbb{R}^n \mid Ax \leq b \}$ \hspace{1cm} real feasible points

- $S = \{ x \in \mathbb{Z}^n \mid Ax \leq b \} = P \cap \mathbb{Z}^n$ \hspace{1cm} integer feasible points

- **Basic properties**
  - If $P = \emptyset$, then $S = \emptyset$.
  - If $z_{LP}$ is finite, then $S = \emptyset$ or $z_P \leq z_{LP}$ is finite.
  - If $z_{LP} = \infty$, then $S = \emptyset$ or $z_P = \infty$.

### Integer hull

- $P = \{ x \in \mathbb{R}^n \mid Ax \leq b \}, \ S = \{ x \in \mathbb{Z}^n \mid Ax \leq b \} = P \cap \mathbb{Z}^n$

- $P_I = \text{conv}(S)$ \hspace{1cm} integer hull

- **Theorem**: $P_I$ is again a polyhedron

- Vertices of $P_I$ belong to $S$

- $\max \{ c^T x \mid x \in S \} = \max \{ c^T x \mid x \in \text{conv}(S) \}$

$\Rightarrow$ reduce integer linear optimization to linear optimization?
Cutting planes

$\text{conv}(S)$ is very hard to compute $\Rightarrow$ approximation by cutting planes

- Solve the linear relaxation
  \[ \max \{ c^T x \mid Ax \leq b, x \in \mathbb{R}^n \} \]
  and compute a basic feasible solution $x^*$.

- If $x^* \in \mathbb{Z}^n$, the integer linear program has been solved.

- If $x^* \notin \mathbb{Z}^n$, generate a cutting plane $a^T x \leq \beta$, which is satisfied by all integer points in $P$, but which cuts off the fractional vertex $x^*$ of $P$.

- Add the inequality $a^T x \leq \beta$ to the system $Ax \leq b$ and solve the relaxation again.

References