Polyhedra

- **Hyperplane** $H = \{x \in \mathbb{R}^n \mid a^T x = \beta\}$, $a \in \mathbb{R}^n \setminus \{0\}, \beta \in \mathbb{R}$
- **Closed halfspace** $\overline{H} = \{x \in \mathbb{R}^n \mid a^T x \leq \beta\}$
- **Polyhedron** $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$
- **Polytope** $P = \{x \in \mathbb{R}^n \mid Ax \leq b, l \leq x \leq u\}, l, u \in \mathbb{R}^n$
- **Polyhedral cone** $P = \{x \in \mathbb{R}^n \mid Ax \leq 0\}$

The feasible set

$$P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$$

of a linear optimization problem is a polyhedron.

Vertices, Faces, Facets

- $P \subseteq \overline{H}, H \cap P \neq \emptyset$ (Supporting hyperplane)
- $F = P \cap H$ (Face)
- $\dim(F) = 0$ (Vertex)
- $\dim(F) = 1$ (Edge)
- $\dim(F) = \dim(P) - 1$ (Facet)
- $P$ pointed: $P$ has at least one vertex.

Illustration
Simplex Algorithm: Geometric view

Linear optimization problem

\[
\max \{ c^T x \mid Ax \leq b, x \in \mathbb{R}^n \}
\]  
(LP)

Simplex-Algorithm (Dantzig 1947)

1. Find a vertex of \( P \).
2. Proceed from vertex to vertex along edges of \( P \) such that the objective function \( z = c^T x \) increases.
3. Either a vertex will be reached that is optimal, or an edge will be chosen which goes off to infinity and along which \( z \) is unbounded.

Basic solutions

- \( Ax \leq b, A \in \mathbb{R}^{m \times n}, \text{rank}(A) = n \).
- \( M = \{1, \ldots, m\} \) row indices, \( N = \{1, \ldots, n\} \) column indices
- For \( I \subseteq M, J \subseteq N \) let \( A_{IJ} \) denote the submatrix of \( A \) defined by the rows in \( I \) and the columns in \( J \).
- \( I \subseteq M, |I| = n \) is called a basis of \( A \) if \( A_I = A_{IN} \) is non-singular.
- In this case, \( v = A_I^{-1} b_I \), where \( b_I \) is the subvector of \( b \) defined by the indices in \( I \), is called a basic solution.
- If in addition \( v \) satisfies \( Ax \leq b \), then \( v \) is called a basic feasible solution and \( I \) is called a feasible basis.

Algebraic characterization of vertices

Theorem

For a non-empty polyhedron \( P = \{ x \in \mathbb{R}^n \mid Ax \leq b \} \) the following holds:

1. \( P \) has at least one vertex if and only if \( \text{rank}(A) = n \).
2. A vector \( v \in \mathbb{R}^n \) is a vertex of \( P \) if and only if it is a basic feasible solution of \( Ax \leq b \), for some basis \( I \).
3. If \( \text{rank}(A) = n \), then for any \( c \in \mathbb{R}^n \), either the maximum value of \( z = c^T x \) for \( x \in P \) is attained at a vertex of \( P \) or \( z \) is unbounded on \( P \).

Remark

It follows from (2) that a polyhedron has at most finitely many vertices.

In general, a vertex may be defined by several bases.

Simplex Algorithm: Algebraic version

- Suppose \( \text{rank}(A) = n \) (otherwise apply Gaussian elimination).
- Suppose \( I \) is a feasible basis with corresponding vertex \( v = A_I^{-1} b_I \).
- Compute \( u^T \overset{\text{def}}{=} c^T A_I^{-1} \) (vector of \( n \) components indexed by \( I \)).
- If \( u \geq 0 \), then \( v \) is an optimal solution, because for each feasible solution \( x \)

\[
c^T x = u^T A_I x \leq u^T b_I = u^T A_I v = c^T v.
\]