Polyhedra

- *Hyperplane* $H = \{x \in \mathbb{R}^n \mid a^T x = \beta\}$, $a \in \mathbb{R}^n \setminus \{0\}, \beta \in \mathbb{R}$
- *Closed halfspace* $\overline{H} = \{x \in \mathbb{R}^n \mid a^T x \leq \beta\}$
- *Polyhedron* $P = \{x \in \mathbb{R}^n \mid Ax \leq b\}, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$
- *Polytope* $P = \{x \in \mathbb{R}^n \mid Ax \leq b,\ l \leq x \leq u\}, l, u \in \mathbb{R}^n$
- *Polyhedral cone* $P = \{x \in \mathbb{R}^n \mid Ax \leq 0\}$

The feasible set

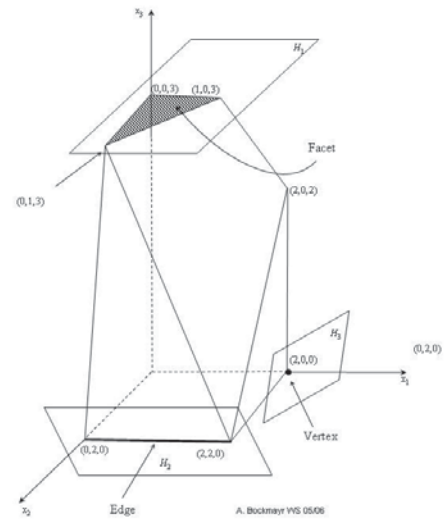
$$P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$$

of a linear optimization problem is a polyhedron.

Vertices, Faces, Facets

- $P \subseteq \overline{H}, H \cap P \neq \emptyset$ *(Supporting hyperplane)*
- $F = P \cap H$ *(Face)*
- $\dim(F) = 0$ *(Vertex)*
- $\dim(F) = 1$ *(Edge)*
- $\dim(F) = \dim(P) - 1$ *(Facet)*
- *P pointed:* P has at least one vertex.

Illustration



Simplex Algorithm: Geometric view

Linear optimization problem

$$\max\{c^T x \mid Ax \le b, x \in \mathbb{R}^n\} \qquad \text{(LP)}$$

Simplex-Algorithm (Dantzig 1947)

1. Find a vertex of P.
2. Proceed from vertex to vertex along edges of P such that the objective function $z = c^T x$ increases.
3. Either a vertex will be reached that is optimal, or an edge will be chosen which goes off to infinity and along which z is unbounded.

Basic solutions

- $Ax \le b,\ A \in \mathbb{R}^{m\times n}, rank(A) = n$.
- $M = \{1, \dots, m\}$ row indices, $N = \{1, \dots, n\}$ column indices
- For $I \subseteq M, J \subseteq N$ let A_{IJ} denote the submatrix of A defined by the rows in I and the columns in J.
- $I \subseteq M, |I| = n$ is called a *basis of A* iff $A_{I*} = A_{IN}$ is non-singular.
- In this case, $v = A_{I*}^{-1} b_I$, where b_I is the subvector of b defined by the indices in I, is called a *basic solution.*
- If in addition v satisfies $Ax \le b$, then v is called a *basic feasible solution* and I is called a *feasible basis.*

Algebraic characterization of vertices

Theorem

For a non-empty polyhedron $P = \{x \in \mathbb{R}^n \mid Ax \le b\}$ the following holds:

1. P has at least one vertex if and only if $rank(A) = n$.
2. A vector $v \in \mathbb{R}^n$ is a vertex of P if and only if it is a basic feasible solution of $Ax \le b$, for some basis I.
3. If $rank(A) = n$, then for any $c \in \mathbb{R}^n$, either the maximum value of $z = c^T x$ for $x \in P$ is attained at a vertex of P or z is unbounded on P.

Remark

It follows from (2) that a polyhedron has at most finitely many vertices.

In general, a vertex may be defined by several bases.

Simplex Algorithm: Algebraic version

- Suppose rank$(A) = n$ (otherwise apply Gaussian elimination).
- Suppose I is a feasible basis with corresponding vertex $v = A_{I*}^{-1} b_I$.
- Compute $u^T \overset{\text{def}}{=} c^T A_{I*}^{-1}$ (vector of n components indexed by I).
- If $u \ge 0$, then v is an optimal solution, because for each feasible solution x

$$c^T x = u^T A_{I*} x \le u^T b_I = u^T A_{I*} v = c^T v.$$