Linear programming

Optimization Problems

- **General optimization problem**

  $$\max \{ z(x) \mid f_j(x) \leq 0, x \in D \} \text{ or } \min \{ z(x) \mid f_j(x) \leq 0, x \in D \}$$

  where $D \subseteq \mathbb{R}^n, f_j : D \rightarrow \mathbb{R},$ for $j = 1, \ldots, m, z : D \rightarrow \mathbb{R}$.

- **Linear optimization problem**

  $$\max \{ c^T x \mid Ax \leq b, x \in \mathbb{R}^n \}, \text{ with } c \in \mathbb{R}^n, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$$

- **Integer optimization problem**: $x \in \mathbb{Z}^n$

- **0-1 optimization problem**: $x \in \{0, 1\}^n$

Feasible and optimal solutions

- Consider the optimization problem

  $$\max \{ z(x) \mid f_j(x) \leq 0, x \in D, j = 1, \ldots, m \}$$

- A feasible solution is a vector $x^* \in D \subseteq \mathbb{R}^n$ such that $f_j(x^*) \leq 0$, for all $j = 1, \ldots, m$.

- The set of all feasible solutions is called the feasible region.

- An optimal solution is a feasible solution such that

  $$z(x^*) = \max \{ z(x) \mid f_j(x) \leq 0, x \in D, j = 1, \ldots, m \}.$$

- Feasible or optimal solutions
  - need not exist,
  - need not be unique.

Transformations

- $\min \{ z(x) \mid x \in S \} = - \max \{ -z(x) \mid x \in S \}$.

- $f(x) \geq a$ if and only if $-f(x) \leq -a$.

- $f(x) = a$ if and only if $f(x) \leq a \land -f(x) \leq -a$.

Lemma

Any linear programming problem can be brought to the form

$$\max \{ c^T x \mid Ax \leq b \} \text{ or } \max \{ c^T x \mid Ax = b, x \geq 0 \}.$$

Proof:

a) $a^T x \leq \beta \iff a^T x + x' = \beta, x' \geq 0$ (slack variable)

b) $x$ free $\iff x = x^+ - x^-, x^+, x^- \geq 0$.

Practical problem solving

1. Model building
2. Model solving
3. Model analysis

Example: Production problem

A firm produces $n$ different goods using $m$ different raw materials.

- $b_i$: available amount of the $i$-th raw material
- $a_{ij}$: number of units of the $i$-th material needed to produce one unit of the $j$-th good
- $c_j$: revenue for one unit of the $j$-th good.

Decide how much of each good to produce in order to maximize the total revenue \( \rightarrow \) decision variables $x_j$.

Linear programming formulation

\[
\max \quad c_1 x_1 + \cdots + c_n x_n \\
\text{w.r.t.} \quad a_{11} x_1 + \cdots + a_{1n} x_n \leq b_1, \\
\vdots \\
a_{m1} x_1 + \cdots + a_{mn} x_n \leq b_m, \\
x_1, \ldots, x_n \geq 0.
\]

In matrix notation:

\[
\max \{ c^T x \mid Ax \leq b, x \geq 0 \},
\]

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n, x \in \mathbb{R}^n$.

Geometric illustration

\[
\max \quad x_1 + x_2 \\
\text{w.r.t.} \quad x_1 + 2x_2 \leq 3 \\
2x_1 + x_2 \leq 3 \\
x_1, x_2 \geq 0
\]