Algorithms  
WS 2014/15  
Exercises 5

1. ”'sparse'” skip list (Niveau I)  
Each node in the skip list has up to two incoming directed edges pointing to other nodes in the skip list.

(a) Which edges are really necessary for a search and which can be removed?
(b) Can you give a rough estimate for the expected number of edges that can be removed?

2. Skip lists (Niveau II)  
Proof that the height of a skip list has expected value $O(\log n)$ with high probability, i.e. show that the probability that the height deviates from $\log n$ by a large factor is very low.
Hint: You do not need Chernoff bounds or Markov’s inequality to show this.

3. Expected values (Niveau I)  
Let $X$ and $Y$ be random variables:

(a) Prove that $E(X + Y) = E(X) + E(Y)$.
(b) Assume that $X$ and $Y$ are independent. Prove that $E(XY) = E(X)E(Y)$
(c) Assume that $X$ takes values $\{0, 1, 2, \ldots\}$. Show that $E(X) = \sum_{k=1}^{\infty} Pr(X \geq k)$.

4. Tree decomposition (Niveau I)  
Prove the following theorem:  
Let $G = (V, E)$ be a graph, $T$ be a tree decomposition of $G$, and $(x, y)$ an edge in $T$. The deletion of $(x, y)$ divides $T$ into two components $X$ and $Y$. Let $V_x$ and $V_y$ be the ‘pieces’ of $x$ and $y$, respectively. Then deleting the set $V_x \cap V_y$ from $V$ disconnects $G$ into the two subgraphs $G_X - (V_x \cap V_y)$ and $G_Y - (V_x \cap V_y)$. ($G_M$ for $M = X, Y$ is the subgraph of $G$ that consists of all nodes in the ‘pieces’ of $M$.)