1. Tree decomposition (Niveau I)
How large is the largest piece of any tree decomposition for a graph $G$ of $n$ nodes if $G$ is a clique? Prove your answer.

2. Tree decomposition (Niveau I)
Prove the following theorem:
Let $G = (V, E)$ be a graph, $T$ be a tree decomposition of $G$, and $(x, y)$ an edge in $T$. The deletion of $(x, y)$ divides $T$ into two components $X$ and $Y$. Let $V_x$ and $V_y$ be the ‘pieces’ of $x$ and $y$, respectively. Then deleting the set $V_x \cap V_y$ from $V$ disconnects $G$ into the two subgraphs $G_X - (V_x \cap V_y)$ and $G_Y - (V_x \cap V_y)$.
($G_M$ for $M = X, Y$ is the subgraph of $G$ that consists of all nodes in the ‘pieces’ of $M$.)

3. Tree decomposition (Niveau II)
Prove the following theorem:
If graph $G$ contains a $(w + 1)$-linked set of size at least $3w$, then $G$ has tree-width at least $w$.
Suppose, by way of contradiction, that $G$ has a $(w + 1)$-linked set $X$ of size at least $3w$, and it also has a nonredundant TD $(T; \{V_i\})$ of width less than $w$. The idea of the proof is to find a piece $V_i$ that is “centered” with respect to $X$, so that when some part of $V_i$ is deleted from $G$, one small subset of $X$ is separated from another.
4. Tree decomposition (Niveau I)

Use the algorithm presented in the lecture to compute a tree decomposition of the graph below. Use $w = 4$: