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Algorithms

WS 2012/13

Exercises 4

1. **Skip lists (Niveau I)**

   Compute the expected value for the height (h), search time and space consumption if the probability p for each coin flip to produce a 1 is 1/3.

2. **"'sparse"' skip list (Niveau I)**

   Each node in the skip list has up to two incoming directed edges pointing to other nodes in the skip list.

   (a) Which edges are really necessary for a search and which can be removed?

   (b) Can you give a rough estimate for the expected number of edges that can be removed?

3. **Skip lists (Niveau II)** Proof that the height of a skip list has expected value $O(logn)$ with high probability.

   Hint: You do not need Chernoff bounds or Markov's inequality to show this.

4. **Independencies**

Random variables $(X_i)_{i\geq 1}$ are called *pairwise independent* if for all $1 \leq i < j$ and all r_i and r_j holds:

$$Pr(X_i = r_i \wedge X_j = r_j) = Pr(X_i = r_i) \cdot Pr(X_j = s_j)$$

Random variables are called *mutual independent* if for all $n \geq 2$ and all $1 \leq i_1 < i_2 < \ldots < i_n$ and $r_1, r_2, \ldots, r_n$ holds:

$$Pr(\bigwedge_{k=1}^{n}(X_{i_k} = r_k)) = \prod_{k=1}^{n} Pr(X_{i_k} = r_k)$$

(a) Let X and Y be random variables:

i. Prove that $E(X+Y) = E(X) + E(Y)$.

ii. Assume that X and Y are independent. Prove that $E(XY) = E(X)E(Y)$

(b) Given the sample space:

$$U = \{(123), (132), (213), (231), (312), (321), (111), (222), (333)\}$$

We choose a random element u in U. Let X_i the digit in u at position i (for $i = 1, 2, 3$), e.g. $X_3 = 2$ for $u = (312)$. Let N the random variable that equals X_2. Prove:

i. $\forall i, r : 1 \leq i \leq 3, 1 \leq r \leq 3$ gilt: $Pr(X_i = r) = \frac{1}{3}$.

ii. X_1, X_2, and X_3 pairwise independent.

iii. X_1, X_2, and X_3 are not mutual independent.

iv. $E(N) = 2$.

v. $\sum_{i=1}^{E(N)} E(X_i) = 4$.

vi. $E(\sum_{i=1}^{N} X_i) \neq \sum_{i=1}^{E(N)} E(X_i)$.