Polynomial reductions

- A polynomial reduction of $L_1 \subseteq \Sigma_1^*$ to $L_2 \subseteq \Sigma_2^*$ is a polynomially computable function $f : \Sigma_1^* \rightarrow \Sigma_2^*$ with $w \in L_1 \iff f(w) \in L_2$.

- **Proposition.** If $L_1$ is polynomially reducible to $L_2$, then
  1. $L_1 \in P$ if $L_2 \in P$ and $L_1 \in NP$ if $L_2 \in NP$
  2. $L_2 \not\in P$ if $L_1 \not\in P$ and $L_2 \not\in NP$ if $L_1 \not\in NP$.

- $L_1$ and $L_2$ are polynomially equivalent if they are polynomially reducible to each other.

NP-complete problems

- A language $L \subseteq \Sigma^*$ is NP-complete if
  1. $L \in NP$
  2. Any $L' \in NP$ is polynomially reducible to $L$.

- **Proposition.** If $L$ is NP-complete and $L \in P$, then $P = NP$.

- **Corollary.** If $L$ is NP-complete and $P \neq NP$, then there exists no polynomial algorithm for $L$.

Structure of the class NP

![Structure of the class NP](image)

Fundamental open problem: $P \neq NP$?

Proving NP-completeness

- **Theorem** (Cook 1971). SAT is NP-complete.

- **Proposition.** $L$ is NP-complete if
  1. $L \in NP$
  2. there exists an NP-complete problem $L'$ that is polynomially reducible to $L$.

- **Example:** INDEPENDENT SET
  
  Instance: Graph $G = (V, E)$ and $k \in \mathbb{N}, k \leq |V|$.
  
  Question: Is there a subset $V' \subseteq V$ such that $|V'| \geq k$ and no two vertices in $V'$ are joined by an edge in $E$?
Reducing 3SAT to INDEPENDENT SET

- Let $F$ be a conjunction of $n$ clauses of length 3, i.e., a disjunction of 3 propositional variables or their negation.
- Construct a graph $G$ with $3n$ vertices that correspond to the variables in $F$.
- For any clause in $F$, connect by three edges the corresponding vertices in $G$.
- Connect all pairs of vertices corresponding to a variable $x$ and its negation $\neg x$.
- $F$ is satisfiable if and only if $G$ contains an independent set of size $n$.

NP-hard problems

- *Decision problem*: solution is either yes or no
- Example: Traveling salesman decision problem:
  Given a network of cities, distances, and a number $B$, does there exist a tour with length $\leq B$?
- *Search problem*: find an object with required properties
- Example: Traveling salesman optimization problem:
  Given a network of cities and distances, find a shortest tour.
- Decision problem $NP$-complete $\Rightarrow$ search problem $NP$-hard
- $NP$-hard problems: at least as hard as $NP$-complete problems

NP-hard problems in bioinformatics

Sperschneider 08

- Multiple alignment
- Shortest common superstring
- Protein threading
- Pseudoknot prediction
- Bi-Clustering
- ...

Further complexity classes

$coNP$: Problems whose complement is in $NP$

$PSPACE$: Problems solvable in polynomial space

$EXPTIME$: Problems solvable in exponential time
Literature

- J. E. Hopcroft and J. D. Ullman: Introduction to automata theory, languages and computation. Addison-Wesley, 1979
- C. H. Papadimitriou: Computational complexity. Addison-Wesley, 1994