1. Overview

This section is adapted from Illusie [13]. First notation: Let $A$ be a complete discrete valuation ring of perfect residue field $k$ of characteristic $p > 0$ and field of fractions $K$ of characteristic 0, $K_0$ be the fraction field of $W = W(k)$, the Witt ring of $k$, $e = [K : K_0]$, the absolute ramification index of $A$ and $\bar{K}$ is the algebraic closure of $K$ with $G = Gal(\bar{K}/K)$. Let $\mathcal{C}_K$ (resp. $\mathcal{C}_k$) denote the category of proper and smooth schemes over $\bar{K}$ (resp. $k$).

For $X_K \in \mathcal{C}_K$, the de Rham cohomology

$$H^*_{DR}(X_K/K) = H^*(X_K, \Omega^*_{X_K/K})$$

is equipped with an extra structure of the Hodge Filtration due to the degeneration of the Hodge -de Rham spectral sequence at $E_1$. Assume moreover that there exist a smooth and proper scheme over $A$ such that

\begin{footnotesize}
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$X \otimes K = X_K$. (This is the case of good restriction, we will be working in this case for the seminar even though it is a very strong assumption and we do have a theory for the case of semi-stable reduction via log-crystalline cohomology, for example see [13].) Let $X_k = X \otimes_k k$ be the special fiber, thus, $X_k \in \mathcal{C}_k$ and we can consider its crystalline cohomology $H^*(X/W)$, a finitely generated graded $W$-module, endowed with the $\sigma$-linear Frobenius operator $\phi$ (where $\sigma$ is the Frobenius of the Witt ring), such that $\phi \otimes \mathbb{Q}$ is bijective. When $e \leq p - 1$, thereby we have a divided power structure on the maximal ideal of $A$ (see [2]), there is a canonical isomorphism

$$H^\text{crys}_{X_k/W} \otimes W A \xrightarrow{\sim} H^*_D(X/A) := H^*(X, \Omega^\bullet_{X/A}),$$

but in the case of $e > p - 1$, we only have the following isomorphism due to Berthelot-Ogus [3]

$$H^\text{crys}_{X_k/W} \otimes W K \xrightarrow{\sim} H^*_D(X/A) \otimes W K = H^*_D(X_K/K).$$

This gives us a $K_0$ structure on $H^*_D(X_K/K)$ via

$$H^*_0(X_K) := H^\text{crys}_{X_k/W} \otimes W K_0,$$

endowed with the $\sigma$-linear automorphism $\phi$ defined by the Frobenius.

We denote these extra structures on $H^*_D(X_K/K)$ as the tuple

$$(\text{Fil}^*, H^*_0(X_K), \phi).$$

This makes $H^*_D(X_K/K)$ an object in the category of filtered $\phi$-modules over $K$ denoted as $\text{MF}_K(\phi)$, as defined by Fontaine [7].

Now for $X_K \in \mathcal{C}_K$, note that $G = \text{Gal} \bar{K}/K$ acts on the $p$-adic étale cohomology $H^*(X_K, \mathbb{Q}_p)$. The $p$-adic Hodge theory compares this representation with the de Rham cohomology considered as a filtered $\phi$-module over $K$. For the case of good reduction, one gets the following period isomorphisms involving the Barsott-Tate period rings $B_{\text{cris}} \subset B_{\text{DR}}$ and $B_{\text{HT}}$ constructed by Fontaine [7].

Our aim for this seminar is to understand the proof of the following period isomorphisms conjectured by Fontaine and proven by Faltings:

**The Hodge Tate Decomposition**

Let $C := \bar{K}$. Then $C$ is algebraically closed and $G$ acts continuously on it. Then Tate [22] showed that

$$(1) \quad H^0(G, C(i)) = \begin{cases} K & \text{if } i = 0; \\ 0 & \text{if } i \neq 0, \end{cases}$$

where $(i)$ is the usual Tate twist by the $i$-th power of the cyclotomic character.
Theorem 1 (Faltings [5]). For $X_K \in \mathcal{C}_K$, there exists a natural $G$-equivariant isomorphism
\begin{equation}
\oplus (C \otimes_K H^{m-i}(X_K, \Omega^i)) \sim C \otimes_{\mathbb{Q}_p} H^m(X_{\overline{K}}, \mathbb{Q}_p),
\end{equation}
where $\Omega^i = \Omega^i_{X_K/K}$ and $G$ acts diagonally on the right hand side.

Remark 2. (1) Note that no good reduction assumption is required for this theorem.
(2) There is another proof of this theorem by Scholze in the setting of perfectoid spaces [21].
(3) There is also a proof by Niziol via K-theory [15] and Bellinson via derived de Rham (see here).

From the above theorem using 1 one deduces that the $p$-adic representation $H^m(X_{\overline{K}}, \mathbb{Q}_p)$ determines the Hodge numbers as follows:
\[ h^{i,m-i} := \dim H^{m-1}(X_K, \Omega^i) = \dim_K (C \otimes_{\mathbb{Q}_p} H^m(X_K, \mathbb{Q}_p)(i))^G. \]

Now defining the Hodge-Tate ring of periods as
\[ B_{HT} := \bigoplus_{i \in \mathbb{Z}} C(i) \]
with its natural $G$-action, one can rewrite the Hodge-Tate decomposition in the form of a $G$-equivariant isomorphism
\begin{equation}
B_{HT} \otimes_K \bigoplus H^{m-i}(X_K, \Omega^i) \sim B_{HT} \otimes_{\mathbb{Q}_p} H^m(X_{\overline{K}}, \mathbb{Q}_p),
\end{equation}
compatible with the natural gradings on both sides ($C(j) \otimes H^{m-i}(X_K, \Omega^i)$ being of degree $i+j$). Then the Hodge cohomology can be recovered from the $p$-adic étale cohomology by
\[ \bigoplus_i H^{m-i}(X_K, \Omega^i) \sim (B_{HT} \otimes_{\mathbb{Q}_p} H^m(X_{\overline{K}}, \mathbb{Q}_p))^G. \]

The $C_{DR}$ conjecture
Now for recovering the de-Rham cohomology, we will need the $B_{DR}$ ring of periods, which is a $K$-algebra that is a complete discrete valuation field, having an action of $G$ and whose associated graded algebra for the filtration defined by the valuation is $B_{HT}$, with its $G$-action.

Theorem 3 (Faltings [6]). For $X_K \in \mathcal{C}_K$, there exists a natural $G$-equivariant isomorphism
\begin{equation}
B_{DR} \otimes_K H^m_{DR}(X_{K}/K) \sim B_{DR} \otimes_{\mathbb{Q}_p} H^m(X_K, \mathbb{Q}_p),
\end{equation}
compatible with the filtrations.

Again using 1 $(B_{DR})^G = K$, and therefore $H^m_{DR}(X_{K}/K)$, a filtered $K$-filtered module is recovered as
\[ H^m_{DR}(X_{K}/K) \sim (B_{DR} \otimes_{\mathbb{Q}_p} H^m(X_K, \mathbb{Q}_p))^G. \]
However we cannot recover the $p$-adic étale cohomology yet as a Galois representation, with this period isomorphism, here we will need more assumption, for example, having good reduction.In the case of good
reduction, we need the ring $B_{\text{cris}}$ which is a sub-$G-K_0$-algebra of $B_{\text{DR}}$, such that $K \otimes_{K_0} B_{\text{cris}} \to B_{\text{DR}}$ is injective and induces an isomorphism on the associated graded objects, $K \otimes_{K_0} B_{\text{cris}}$ being endowed with the induced filtration. In addition, $B_{\text{cris}}$ has a $\sigma-K_0$-linear endomorphism $\phi$, and

\[(5) \quad \mathbb{Q}_p = \{ x \in B_{\text{cris}} | 1 \otimes x \in \text{Fil}^0 B_{\text{DR}} \text{ and } \phi(x) = x \}.
\]

Moreover, $B_{\text{cris}}^G = K_0$.

The $C_{\text{cris}}$ conjecture

**Theorem 4** (Faltings \[6\]). For $X/A$ proper and smooth, there exists a natural isomorphism

$$B_{\text{cris}} \otimes_{K_0} H^m_0(X_K) \sim \to B_{\text{cris}} \otimes \mathbb{Q}_p H^m(X_{\bar{K}}, \mathbb{Q}_p),$$

compatible with the actions of $\phi$ and $G$ on both the sides, as well as the filtrations after extension of scalars to $K$.

Again, we have

\[(6) \quad H^m_0(X_K) \sim \to (B_{\text{cris}} \otimes \mathbb{Q}_p H^m(X_{\bar{K}}, \mathbb{Q}_p))^G,
\]

and the Hodge filtration on $K \otimes_{K_0} H^m_0(X_K) \sim \to H^m_{\text{DR}}(X_K/K)$ is obtained by a similar formula, thus $H^m_{\text{DR}}(X_K/K)$ as an object $MF_K(\phi)$ is determined by the the Galois representation $H^m(X_{\bar{K}}, \mathbb{Q}_p)$. But conversely as well, so $H^m(X_{\bar{K}}, \mathbb{Q}_p)$ is determined:

\[(7) \quad H^m(X_{\bar{K}}, \mathbb{Q}_p) \sim \to \{ x \in B_{\text{cris}} \otimes H^m_0(X_K) | \phi(x) = x, 1 \otimes x \in \text{Fil}^0 \} \text{ as representations of } G.
\]

We will mainly follow the faltings approach and prove the comparison theorems via almost mathematics developed by Faltings. We will then end the seminar by introducing the theory of $p$-adic Galois representations (implicitly).

2. The Plan

We will prove the $C_{\text{cris}}$ conjecture in the seminar and the Hodge-Tate decomposition only for the good reduction case $+ \epsilon$. We use \[17\] as the main source and will refer to other references when and where needed.

2.1. Almost Mathematics and Purity Theorem, 24.04.17. Do section 2, of olsson \[17\], and sketch a proof of purity theorem following Falting Thm 3.1 and Thm 3.2 chapter 1, \[5\].

**Detailed description**: Do 2.1 \[17\], remark that in particular $m^2 = m$. Then do example 2.2, \[17\] and example 2.1.2, 2.1.3 from \[9\]. Explain section 2.3 (if time permits give a brief review of localized category either following Gabriel \[10\] chapter 3, section 1 or following \[19\] section
4.3, here dense subcategory means Serre subcategory). Do all the paragraphs from definition 2.4 till the statement of 2.17, the almost purity theorem. Sketch a proof of this theorem following Falting’s Thm 3.1 and Thm 3.2 chapter 1, [5]. Then just introduce the set up of 2.19 with Remark 2.20 and only state the lemmas 2.21 to 2.23. Preferably combine them into one lemma.

2.2. Galois Cohomology, 08.05.17. Section 3 upto corollary 3.14 [17]. This talk involves a lot of computations in group cohomology. Prepare the audience by making a very short recap of some facts and definitions in group cohomology. You can consult Chapter 3 of [11] or have a look at [20]. Focus mostly on facts that are used in proofs of section 3 of [17] and don’t spend too much time on that part. Then proceed to the main part, i.e. [17, 3]. The main results are lemma 3.10 and the discussion below it, i.e. 3.11. Skip whole subsection 3.6. You might also consider skipping the proof of proposition 3.5. Besides this, try to give details of proofs from 3.1 to 3.11. State 3.12 and 3.14.

2.3. Galois cohomology II and Logarithmic geometry, 15.05.17. Do the remaining section 3 and section 4 [17]. Also explain the definition of log structure.

Detailed description: Start by explaining paragraph 3.15, [17] to give a proof of 3.16, then do 3.17 and prove 3.19 and add remark 3.20 in the statement of 3.19 and then combine the following paragraph to give a sketch of proof of Theorem 3.25. After this explain briefly the section 4 [17]. Start with the definition of log structure using either [14] or see here or [16] or Lectures on Logarithmic Algebraic Geometry Chapter 4. Explain the log structure and a trivial log structure on a scheme. Explain paragraphs 4.1-4.6 and sketch a proof of thm 4.7 and in the end prove 4.9, 4.10, 4.11.

2.4. The $K(\pi,1)$, 22.05.17. Section 5 of [17]. Detailed description: Present results of Section 5. Make sure to explain remark 5.2 which is important. The main result is theorem 5.4. Try to give the details in proofs of lemma 5.5, example 5.9 and corollary 5.10, while you might be sketchy in proofs of Lemma 5.7, 5.8, and theorem 5.11.

2.5. The topos $\mathfrak{X}_K^0$ and $\hat{\mathfrak{X}}_K^0$, 29.05.17. Do section 6 and 9 of [17].

Detailed description: Do 6.1, along with section 9.1, state lemma 6.2, remark 6.3, 6.4 (Compare from Olsson [18] Example 2.2.33, Proposition 2.2.31.) Continue with lemma 6.5, 6.6, then go to do 9.2, state 9.3 and 9.4. (do proofs if time permits). Then go back to 6.7 (compare from stacks project: simplicial spaces tag 09VI). Then do 6.8, state Proposition 6.9 with only a sketch of the proof. Then go on to prove corollary 6.11, 6.13, 6.14 and 6.15. Then state 6.16 with remark 6.17.
2.6. **More on proof of Theorem 6.16** **12.06.17.** Sketch the proof of Theorem 6.16 following section 7 and 8 of [18].

**Detailed description:** Explain the main results of section 7 [18], which are 7.13, 7.26. Then go to section 8, starting with 8.1 and explain briefly how to reduce the proof of 6.16 to the case when $D$ has simple normal crossing divisors. Then do 8.3, 8.4, lemma 8.5 (with proof). State theorem 8.9 along with the extra properties as the compatibility with cup product and Poincare duality (don’t prove them). Then explain the construction of 8.15 and prove 8.16. Do 8.17, 8.18, 8.19, 8.20, 8.21 (sketch a proof with prop. 8.23) (which finishes at 8.54, if time permits explain briefly the trick after the corollary), finish the sketch of proof theorem 6.16 and then state the result 8.60 and 8.61. (Seeing a proof would be nice.)

2.7. **Ring of Periods, 19.06.17.** section 11 of [17].

**Detailed description:** Explain the construction of 11.1 [17]. Do 11.2, state the lemma 11.3, explain paragraph 11.4 but skip 11.4.3 and state the lemmas 11.5, 11.6, 11.8 and 11.9 without proof. Then do paragraph 11.10, lemma 11.11 without proof, 11.12 - 11.19, and then the lemma 11.20, corollary 11.21, lemmas 11.22-11.24 and corollary 11.25 with sketch of proofs, if time permits.

2.8. **More Galois Cohomology and cohomological computations, 26.06.17.** Do section 10 and 12 [17].

**Detailed description:** Start with section 10 of [17]. Be reasonably quick with 10.1 - 10.3, as there isn’t much going on. Then explain more carefully 10.4. and 10.5. Proceed to section 12. Present 12.1 - 12.5 with proofs. Then state the results from 12.6 to 12.11 with only sketching or skipping the proofs, depending on time.

2.9. **Crystalline Cohomology and crystalline sheaves, 03.07.17.**

Do section 13 of [17]. This is crystalline cohomology via convergent topos, we do it in logarithmic setting and we introduce the category of filtered $\phi$ modules.


2.10. **Comparison isomorphisms I, 10.07.17.** Do section 14 [17].

**Detailed description:** Recall the theorem 13.21 which we are going to prove and which is basically the main theorem of the whole paper. Present the results 14.1-14.11 giving a reasonable amount of details.
2.11. **Crystalline comparison theorem.**

**Detailed description:** List quickly the definitions of the rings we already know (5.1) and define new ones (5.2). State lemma 15.3 (the proof is optional, depending on time). Then explain 15.4. Now state and prove the main result of this section, i.e. theorem 15.5. Sum up with corollary 15.9.

**Remarks for the speakers**

1. In case you cannot find any reference, please contact the organizers. We will have most of the references in the printed form or the Books in the office 111.
2. If you would like to use a different reference, please discuss with one of the organizers beforehand.
3. We will also add additional material on the website later to help with the talks which have long proofs like Almost purity and comparison theorem 6.16.

**References**


