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Exercise 6 for Number theory III[1]

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Exercise 6.1. Let K be a finite extension of $\mathbb{Q}_p$. Recall that we equip $K^\times$ with the topology for which $U_K^{(n)} = 1+\mathfrak{m}_K^n$, $n \geq 1$, is a fundamental system of open neighborhoods of 1.

(1) Show that $(K^\times)^n = \{a^n \mid a \in K^\times\}$ is an open subgroup of finite index in $K^\times$, for all $n \geq 1$. (*Hint:* First show that for all $n \geq 1$ there are $i, j \geq 1$ with $\mathfrak{m}_K^j = n\mathfrak{m}_K^i$ and such that the exponential function from Exercise 5.2 is defined on $\mathfrak{m}_K^i$. Use the properties of exp to conclude that $(K^\times)^n$ is open. Use Exercise 4.2 to show that the index is finite.)
(2) Show that any subgroup of finite index of $K^\times$ is open.

Exercise 6.2. Let K be a local field which is not $\mathbb{R}$ or $\mathbb{C}$. Use the Main Theorem of Local Class Field Theory to show that the finite unramified extensions of K are in one-to one correspondence with subgroups of finite index in $K^\times/\mathcal{O}_K^\times \cong \mathbb{Z}$.

Exercise 6.3. Let $p \neq 2$ be a prime number. Show that there are exactly $p+1$ abelian extensions of degree p over $\mathbb{Q}_p$ only one of which is unramified. To this end proceed as follows:

(1) Use the Main Theorem of Local Class Field Theory and Exercise 6.1 to show that the number of abelian extensions of degree p over $\mathbb{Q}_p$ is equal to the number of subgroups of $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^p$.
(2) Show that $\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^p \cong \mathbb{Z}/p \times \mathbb{Z}/p$.
(3) Count the subgroups of index p in $\mathbb{Z}/p \times \mathbb{Z}/p$.
(4) Use Exercise 6.2 to show that there is only one unramified abelian extension of degree p over $\mathbb{Q}_p$.

Exercise 6.4. Let K be a local field, which is not $\mathbb{R}$ or $\mathbb{C}$. Let $k = \mathbb{F}_q$ be its residue field. Recall from the lecture that we have the exact sequence

$$0 \to I(K^{\mathrm{ab}}/K) \to G(K^{\mathrm{ab}}/K) \to G(k) \to 0,$$

[1]This exercise sheet will be discussed on November 28. If you have questions or remarks please contact `kay.ruelling@fu-berlin.de` or `kindler@math.fu-berlin.de` or `l.zhang@fu-berlin.de`

where $I(K^{\mathrm{ab}}/K)$ denotes the inertia group and $G(k) \cong \hat{\mathbb{Z}}$ the absolute Galois group of k. We view $\mathbb{Z}$ as a subgroup of $G(k)$ by mapping $1 \in \mathbb{Z}$ to the q-power Frobenius. Denote by d the composition $d : G(K^{\mathrm{ab}}/K) \to G(K^{\mathrm{ur}}/K) \cong G(k)$. We define the *abelian Weil group of* K to be

$$W^{\mathrm{ab}}(K) := d^{-1}(\mathbb{Z}).$$

It clearly contains $I(K^{\mathrm{ab}}/K)$ and we equip it with the topology for which the open subsets of $I(K^{\mathrm{ab}}/K)$ form a fundamental system of open neighborhoods of 1.

Show that the local Artin map $\rho_K : K^\times \to G(K^{\mathrm{ab}}/K)$ induces an isomorphism of topological groups

$$\rho_K : K^\times \xrightarrow{\simeq} W^{\mathrm{ab}}(K)$$

under which $\mathcal{O}_K^\times$ is mapped isomorphically to $I(K^{\mathrm{ab}}/K)$.

(*Hint:* Deduce this from the Main Theorem of Local Class Field Theory using that $\mathcal{O}_K^\times$ and $I(K^{\mathrm{ab}}/K)$ are both complete.)