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# Exercise 5 for Number theory III[1]

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**Exercise 5.1.** Let $K$ be a complete discrete valuation field with perfect residue field $k$ (i.e. $K = \mathrm{Frac}(A)$ with $A$ a complete DVR and $k = A/\mathfrak{m}$ is perfect) and fix a separable closure $K^{\mathrm{sep}}$. Show that the maximal unramified extension $K^{\mathrm{ur}}$ of $K$ in $K^{\mathrm{sep}}$ is a discrete valuation field with residue field $k^{\mathrm{sep}}$ the separable closure of $k$.

**Exercise 5.2.** Let $K$ be a finite extension of $\mathbb{Q}_p$ and denote by $e = e(K/\mathbb{Q}_p)$ its ramification index and by $v_K : K^\times \to \mathbb{Z}$ its normalized discrete valuation (so that $v_K(p) = e$). Denote by $\mathfrak{m}$ the maximal ideal of $\mathcal{O}_K$ and set $U_K^{(n)} := 1 + \mathfrak{m}^n$, $n \geq 1$. Let $\log : K^\times \to K$ be the group homomorphism from Exercise 4.4, (4). Show:

(1) log induces a map $\log : U_K^{(n)} \to \mathfrak{m}^n$, for all $n > \frac{e}{p-1}$. (*Hint:* Show that $v_K(z^j/j) > v_K(z)$, for all $j \geq 2$ and $z \in K$ with $v_K(z) > \frac{e}{p-1}$.)

(2) For $x \in K$ with $v_K(x) > \frac{e}{p-1}$ the series $\sum_{j=0}^{\infty} x^j/j!$ converges, see Exercise 4.4. (*Hint:* If $j \in \mathbb{N}$ is written in the form $j = a_0 + a_1 p + \ldots a_s p^s$, with $a_i \in [0, p-1]$, then $v_K(j!) = \frac{1}{p-1}(j - (a_0 + a_i + \ldots + a_s))$.)

(3) For all $n > \frac{e}{p-1}$ there is a well defined continuous map

$$\exp : \mathfrak{m}^n \to U_K^{(n)}, \quad x \mapsto \exp(x) := \sum_{j=0}^{\infty} \frac{x^j}{j!}.$$

(4) For all $n > \frac{e}{p-1}$ the maps $\log_{|U^{(n)}}$ and $\exp_{|\mathfrak{m}^n}$ are inverse to each other, i.e. we have an isomorphism of topological groups

$$U_K^{(n)} \cong \mathfrak{m}^n.$$

**Exercise 5.3.** (1) Show that up to isomorphism there is a unique unramified quadratic extension of $\mathbb{Q}_3$ and it is isomorphic to $\mathbb{Q}_3(\sqrt{-1})$.

[1]This exercise sheet will be discussed on November 21. If you have questions or remarks please contact `kay.ruelling@fu-berlin.de` or `kindler@math.fu-berlin.de` or `l.zhang@fu-berlin.de`

(2) Show that $\mathbb{Z}_3^\times/(\mathbb{Z}_3^\times)^2 = \{\pm 1\}$.
(3) Show that up to isomorphism there are two ramified quadratic extensions of $\mathbb{Q}_3$ and they are isomorphic to $\mathbb{Q}_3(\sqrt{3})$ or $\mathbb{Q}_3(\sqrt{-3})$.

Thus all together we see that up to isomorphism there are only three quadratic extensions of $\mathbb{Q}_3$.

**Exercise 5.4.** Let $K$ be a local field. Let $p$ be the characteristic of its residue field.

(1) Show that if $\zeta \in \bar{K}$ is an $n$-th root of unity with $(n, p) = 1$, then $K(\zeta)$ is unramified over $K$.
(2) Show that $K^{\mathrm{ur}}$ is obtained by adjoining all $n$-th root of unity with $(n, p) = 1$ to $K$. (*Hint:* One inclusion follows from (1) the other from Hensel's Lemma.)