3. November 2014

Exercise 4 for Number theory III[1]

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Exercise 4.1. Let K be a complete discrete valuation field with normalized discrete valuation $v : K^\times \to \mathbb{Z}$ and L/K a finite separable field extension. We know from the lecture that L is also a complete discrete valuation field. Show that its normalized discrete valuation is given by

$$v_L : L^\times \to \mathbb{Z}, \quad a \mapsto \frac{1}{f} \cdot v(\mathrm{Nm}_{L/K}(a)),$$

where $\mathrm{Nm}_{L/K} : L^\times \to K^\times$ is the norm map and $f = f(L/K)$ is the inertia degree.

(*Hint:* To show $v_L(L^\times) \subset \mathbb{Z}$, let E be the maximal unramified subextension of L/K and use $\mathrm{Nm}_{L/K} = \mathrm{Nm}_{E/K} \circ \mathrm{Nm}_{L/E}$.)

Exercise 4.2. Let K be a local field (not $\mathbb{R}, \mathbb{C}$) and let $q = p^n$ be the cardinality of its residue field. Set $\mu_{q-1}(K) := \{a \in K^\times \mid a^{q-1} = 1\}$.

(1) Show that the natural surjection $\mathcal{O}_K \to \mathbb{F}_q$ induces a bijection of groups $\mu_{q-1}(K) \xrightarrow{\simeq} \mathbb{F}_q^\times \cong \mathbb{Z}/(q-1)\mathbb{Z}$. (*Hint*: Hensel's Lemma.)
(2) Let $\pi \in \mathcal{O}_K$ be a local parameter. Show that the group $K^\times$ admits a canonical decomposition

$$K^\times \cong \pi^{\mathbb{Z}} \times \mu_{q-1}(K) \times U_K^{(1)},$$

where $U_K^{(1)} = 1 + \pi\mathcal{O}_K$.
(3) Show that if $a \in K^\times$ has finite order n (i.e. the group $\{1, a, a^2, \dots\}$ has cardinality n), then $n | q-1$.

Exercise 4.3. Recall that we proved the following in Number Theory 2: Let $\zeta \in \bar{\mathbb{Q}}$ be a p^r-th primitive root of unity. Then

(1) $[\mathbb{Q}(\zeta) : \mathbb{Q}] = \varphi(p^r) := (p-1)p^{r-1}$.
(2) $\mathcal{O}_{\mathbb{Q}(\zeta)} = \mathbb{Z}[\zeta]$.
(3) $p\mathbb{Z}[\zeta] = (1-\zeta)^{\varphi(p^r)}$.

Show:

[1]This exercise sheet will be discussed on November 14. If you have questions or remarks please contact `kay.ruelling@fu-berlin.de` or `kindler@math.fu-berlin.de` or `l.zhang@fu-berlin.de`

(1) The same conclusion holds when we replace $\mathbb{Q}$ by $\mathbb{Q}_p$ and $\mathbb{Z}$ by $\mathbb{Z}_p$.
(2) There is a canonical decomposition

$$\mathbb{Q}_p(\zeta)^\times \cong (1-\zeta)^{\mathbb{Z}} \times \mathbb{Z}/(p-1)\mathbb{Z} \times U^{(1)}_{\mathbb{Q}_p(\zeta)}.$$

Exercise 4.4. Let K be a finite extension of $\mathbb{Q}_p$. We know that it is a complete discrete valuation field. Let $A, \mathfrak{m}, v_K$ be its ring of integers, its maximal ideal and its normalized discrete valuation.

(1) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of elements in K and assume $v_K(a_n) \to \infty$, for $n \to \infty$. Show that the sum $\sum_{n=1}^\infty a_n$ converges, i.e. there exists a unique element $s \in K$ such that $s \equiv \sum_{n=1}^\infty a_n \mod \mathfrak{m}^N$ for all $N \geq 1$. Notice that by assumption the sum is finite modulo $\mathfrak{m}^N$. (In terms of the non-archimedean absolute value $|-|_{v_K}$ defined in Exercise 1.1 one can rephrase this by saying: If $(a_n)_n$ is a null sequence in K with respect to $|-|_{v_K}$ then the sequence $(\sum_{n\geq 1}^N a_n)_N$ converges in K.)
(2) Show that for $x \in \mathfrak{m}$ the sum $\sum_{n=1}^\infty (-1)^{n-1}\frac{x^n}{n}$ converges. We set

$$\log(1+x) := \sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}, \quad x \in \mathfrak{m}.$$

(3) Show that we obtain a continuous group homomorphism

$$\log : U_K^{(1)} \to K, \quad 1+x \mapsto \log(1+x).$$

Here we equip $U_K^{(1)}$ with the topology which is uniquely determined by the property that $U_K^{(1)}$ is a topological group and the sets $U_K^{(n)} := 1 + \mathfrak{m}^n$, $n \geq 1$, form a fundamental system of open neighborhoods of 1 and similar K is the topological group with $\mathfrak{m}^n$, $n \geq 1$, as a fundamental system of open neighborhoods.
(4) Show that there is a continuous homomorphism

$$\log : K^\times \to K,$$

which is uniquely determined by the properties that $\log_{|U_K^{(1)}}$ is the map from (3) and $\log(p) = 0$. (*Hint*: Use Exercise 4.2.)