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Exercise 2 for Number theory III[1]

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Exercise 2.1. Let I be a directed set (i.e. there is a partial ordering $\leq$ on I such that for any two elements $i, j \in I$ there exists $n \in I$ with $n \geq i, j$). We say that a family $(G_i)_{i\in I}$ is a projective system of groups, if the G_i's are groups and for any $j \geq i$ there exists a group $\varphi_{j,i} : G_j \to G_i$ homomorphism such that $\varphi_{i,i} = \mathrm{id}_{G_i}$ and $\varphi_{j,i} \circ \varphi_{k,j} = \varphi_{k,i}$ for all $k \geq j \geq i$. For such a projective system define

$$\varprojlim_{i\in I} G_i := \{(g_i) \in \prod_{i\in I} G_i \,|\, \varphi_{j,i}(g_j) = g_i \text{ for all } j \geq i\}.$$

Show that $\varprojlim_{i\in I} G_i$ has the following universal property: Let H be a group and $h_i : H \to G_i$, $i \in I$, group homomorphisms such that $\varphi_{j,i} \circ h_j = h_i$, for all $j \geq i$. Then there exists a unique morphism $h : H \to \varprojlim_{i\in I} G_i$, which when composed with the natural projection maps $\varprojlim_{i\in I} G_i \to G_j$ equals h_j for all $j \in I$.

Remark: $\varprojlim_{i\in I} G_i$ is called the *projective limit* of the projective system $(G_i)_{i\in I}$. It can be defined in a similar way if we replace groups by rings, modules, etc. (In general it exists by definition in a complete category.)

Exercise 2.2. Let G be a topological group. Show:

(1) Any open subgroup of G is also closed.
(2) Assume G is quasi-compact (i.e. any open cover of G has a finite refinement). Then any open subgroup U of G has finite index in G (i.e. there are only finitely many left cosets of U in G).
(3) Assume G is profinite. Then G is compact (i.e. Hausdorff and quasi-compact) and totally disconnected (i.e. each point $g \in G$ is its own connected component.)

Remark: One can also show that if G is a topological group that is compact and totally disconnected then it is profinite.

[1]This excersise sheet will be discussed on October 31. If you have questions or remarks please contact `kay.ruelling@fu-berlin.de` or `kindler@math.fu-berlin.de`

Exercise 2.3. (1) Let p be a prime number and $\mathbb{F}_q$ be a field with $q = p^r$ Elements. Fix an embedding $\mathbb{F}_q \hookrightarrow \mathbb{F}$ into an algebraic closure.

Show that there is an isomorphism of topological groups $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$.

(*Hint:* You can use that for $n \geq 1$ the Galois group $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ is the cyclic group which is generated by the r-power Frobenius.)

(2) Find a subgroup $H \subset \mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)$ such that $\mathrm{Gal}(\mathbb{F}/\mathbb{F}_q)/H$ is not the Galois group of a Galois extension of $\mathbb{F}_q$. (Notice that by the Galois correspondence this H cannot be a *closed* subgroup.)

(3) Let $\tilde{\mathbb{Q}}$ be the subfield of $\mathbb{C}$ defined by adjoining all n-th roots of unity, $n \in \mathbb{N}$, to $\mathbb{Q}$.

Show that $\tilde{\mathbb{Q}}/\mathbb{Q}$ is Galois and that there is an isomorphism of topological groups $\mathrm{Gal}(\tilde{\mathbb{Q}}/\mathbb{Q}) \cong \hat{\mathbb{Z}}^\times$.

(*Hint:* You can use that the Galois group of the n-th cyclotomic extension of $\mathbb{Q}$ is isomorphic to $(\mathbb{Z}/n\mathbb{Z})^\times$.)

Exercise 2.4. Let G be a profinite group. Denote by G' the closure in G of the subgroup generated by all the commutators $aba^{-1}b^{-1}$, with $a, b \in G$.

(1) Show that G' is a normal subgroup of G. Hence we get a group $G^{\mathrm{ab}} := G/G'$.

(2) Show that the group G^{ab} is abelian and has the following universal property: Any continuous group homomorphism $\varphi : G \to H$ from G into an *abelian* profinite group H factors uniquely via a morphism $\varphi^{\mathrm{ab}} : G^{\mathrm{ab}} \to H$.

(3) Assume $G = \varprojlim_{i \in I} G_i$ (as topological groups), where $(G_i)_{i \in I}$ is a projective system of finite groups. Show

$$G^{\mathrm{ab}} = \varprojlim_{i \in I} G_i^{\mathrm{ab}}.$$