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Exercise 13 for Number theory III[1]

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Exercise 13.1. Let L be a field we define the second Milnor K-theory of L to be

$$K_2^M(L) = L^\times \otimes_{\mathbb{Z}} L^\times / < a \otimes (1-a) \,|\, a \in L \setminus \{0,1\} > .$$

The class of an element $a \otimes b$ in $K_2^M(L)$ is denoted by $\{a,b\}$. Show that the following relations hold:

(1) $\{aa', b\} = \{a,b\} + \{a', b\}$, $\{a, bb'\} = \{a,b\} + \{a,b'\}$, $\{a, 1-a\} = 0$.

(2) $\{a, -a\} = 0$, $\{a,b\} = -\{b,a\}$. (*Hint:* To show the first equality consider the identity $-a = (1-a)/(1-a^{-1})$; the second equality follows from the first.)

Exercise 13.2. Let K be a local field ($\neq \mathbb{C}, \mathbb{R}$). Let $p \neq 2$ be the characteristic of the residue field of K and $n \in \mathbb{N}$ with $(n,p) = 1$. Assume that K contains an n-th primitive root of unity ζ.

(1) Show that for any $a \in K^\times$ there exists a unique continuous homomorphism $\chi_a : G(K^{\rm sep}/K) \to \mathbb{Q}/\mathbb{Z}$ of order $d|n$ which (see Exercise 11.2 (1)) corresponds to $(K(\sqrt[n]{a}), \sigma_a)$, where $K(\sqrt[n]{a})$ denotes the splitting field of $X^n - a$ and $\sigma_a \in G(K(\sqrt[n]{a})/K)$ is the element, which on a root α of $X^n - a$ acts via $\sigma_a(\alpha) = \zeta^{n/d}\alpha$.

(2) Show that for $a, a' \in K^\times$ we have $\chi_{aa'} = \chi_a + \chi_{a'}$.

We define the map

$$(-,-)_{n,K} : K^\times \times K^\times \to \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}, \quad (a,b) \mapsto (a,b)_{n,K} := \mathrm{inv}([A(\chi_a, b)]),$$

where $\mathrm{inv} : \mathrm{Br}(K) \xrightarrow{\simeq} \mathbb{Q}/\mathbb{Z}$ is the isomorphism from the lecture.

(3) Show that $(-,-)_{n,K}$ is $\mathbb{Z}$-bilinear, i.e.

$$(aa', b)_{n,K} = (a,b)_{n,K} + (a',b)_{n,K}, \quad (a, bb')_{n,K} = (a,b)_{n,K} + (a,b')_{n,K}.$$

(4) Show that $(a, 1-a)_{n,K} = 0$. (*Hint:* Let $\alpha \in K(\sqrt[n]{a})$ be a solution of $X^n - a$ and show that $1 - a = 1 - \alpha^n = \prod_{i=0}^{n-1}(1 - \zeta^i \alpha)$ is contained in $\mathrm{Nm}_{K(\sqrt[n]{a})/K}(K(\sqrt[n]{a})^\times)$.)

[1]This exercise sheet will be discussed on January 30. If you have questions or remarks please contact `kay.ruelling@fu-berlin.de` or `kindler@math.fu-berlin.de` or `l.zhang@fu-berlin.de`

(5) Conclude that $(-,-)_{n,K}$ induces a homomorphism

$$h_{n,K} : K_2^M(K)/n \cdot K_2^M(K) \to \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}, \quad \{a,b\} \mapsto (a,b)_{n,K}.$$

(6) Show that if $K(\sqrt[n]{a})$ is unramified of degree $d|n$ over K, then $h_{n,K}(\{a,b\}) = v_K(b)/d$ in $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$.
(7) Conclude from (6) that $h_{n,k}$ is surjective.
(8) Let $\pi \in \mathcal{O}_K$ be a local parameter and $q = p^s$ the cardinality of the residue field. Recall from Exercise 4.2, that we have $K^\times = \pi^{\mathbb{Z}} \times \mu_{q-1}(K) \times U_K^{(1)}$ and that by assumption $n|(q-1)$. Let $\xi \in \mu_{q-1}$ a primitive $(q-1)$th root of unity.

    Show that as a group $K_2^M(K)/nK_2^M(K)$ is generated by the element $\{\pi, \xi\}$ and $\{\xi, -1\}$. (*Hint:* By Hensel's Lemma any element in $U_K^{(1)}$ is an nth power; use the relations from Exercise 13.1.)
(9) Show that $\mu_{q-1}(K) \cap \mathrm{Nm}_{K(\sqrt[n]{\pi})/K}(K(\sqrt[n]{\pi})^\times) \subset (K^\times)^n$.
(10) Assume that $2n|(q-1)$. Show that $\{\xi, -1\} \equiv 0 \bmod nK_2^M(K)$. Conclude that

$$h_{n,K} : K_2^M(K)/nK_2^M(K) \xrightarrow{\simeq} \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}$$

is an isomorphism.

Remark 1. Notice that in the situation above we have $\frac{1}{n}\mathbb{Z}/\mathbb{Z} = \mathrm{Br}(K)[n]$. One can show that under the above assumptions $\mathrm{Br}(K)[n]$ is isomorphic to $H^2(G(k^{\mathrm{sep}}/k), \mu_n(k^{\mathrm{sep}})^{\otimes 2})$ (by Hilbert 90). Thus we obtain an isomorphism

$$h_{n,K} : K_2^M(K)/nK_2^M(K) \xrightarrow{\simeq} H^2(G(k^{\mathrm{sep}}/k), \mu_n(k^{\mathrm{sep}})^{\otimes 2}).$$

It was proven by Suslin-Merkurjev in 1982 that there is such an isomorphism for all fields K and all natural numbers n, which are invertible in $K^\times$.

Exercise 13.3. Let L be a global field and $\mathfrak{p} \subset \mathcal{O}_L$ a maximal prime. Denote by $L_\mathfrak{p}$ the completion of L along $\mathfrak{p}$ and by $\pi \in \mathfrak{p}\mathcal{O}_{L_\mathfrak{p}}$ a local parameter. Let n be a natural number which is prime to the residue characteristic of $L_\mathfrak{p}$. We assume that L contains an n-th primitive root of unity. We define the *n-th power residue symbol* to be the following map

$$\mathcal{O}_L^\times \to \tfrac{1}{n}\mathbb{Z}/\mathbb{Z}, \quad u \mapsto \left(\frac{u}{\mathfrak{p}}\right) := h_{n,K}(\{\pi, u\}),$$

where we use the notation from Exercise 13.2.

(1) Show that
$$\left(\frac{u}{\mathfrak{p}}\right) = 1 \iff u \text{ is an } n\text{th power modulo } \mathfrak{p}.$$

(2) Show that for $L = \mathbb{Q}$, $\mathfrak{p} = p$ an odd prime and $n = 2$, the symbol defined above is equal to the Legendre symbol from Exercise 7.2.