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Exercise 10 for Number theory III[1]

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Exercise 10.1. Let k be a field and A a not necessarily commutative ring with 1 and equipped with an injective ring homomorphism $k \hookrightarrow A$. Show that the following conditions are equivalent:

(1) A is a central simple k-algebra.
(2) There exists a finite Galois extension L/k and an isomorphism $A \otimes_k L \cong M_n(k)$, for some $n \geq 1$.

Exercise 10.2. Let K be a local field (not $\mathbb{R}$, $\mathbb{C}$), denote by $v_K : K^\times \to \mathbb{Z}$ the normalized discrete valuation and by $\mathcal{O}_K$, $\mathfrak{m}$, k the ring of integers, the maximal ideal and the residue field, respectively. Let D be a central division algebra over K and write $[D : K] = n^2$.

(1) For $\alpha \in D \setminus \{0\}$ set

$$v_D(\alpha) := \frac{1}{[K(\alpha) : K]} v_K(\mathrm{Nm}_{K(\alpha)/K}(\alpha)),$$

where $\mathrm{Nm}_{K(\alpha)/K} : K(\alpha)^\times \to K^\times$ is the norm map. Show that this defines a valuation on D extending v_K, i.e. a multiplicative map $v_D : D \setminus \{0\} \to \mathbb{Q}^\times$ with $v_D(\alpha + \beta) \geq \min\{v_D(\alpha), v_D(\beta)\}$ and $v_{D|K} = v_K$.

(2) Show that $v_D(D \setminus \{0\}) \subset \frac{1}{n}\mathbb{Z}$. We set $e := [v_D(D \setminus \{0\}) : \mathbb{Z}]$ and get $e \leq n$.

Set

$$\mathcal{O}_D := \{\alpha \in D \setminus \{0\} \,|\, v_D(\alpha) \geq 0\} \cup \{0\},$$
$$\mathfrak{m}_D := \{\alpha \in D \setminus \{0\} \,|\, v_D(\alpha) > 0\} \cup \{0\}.$$

(3) Show that $\mathcal{O}_D$ is a finite free $\mathcal{O}_K$-module of rank n^2 and $D = \mathcal{O}_D \otimes_{\mathcal{O}_K} K$.
(4) Show that $\mathfrak{m}_D \subset \mathcal{O}_D$ is a two-sided ideal and that $k_D := \mathcal{O}_D/\mathfrak{m}_D$ is a finite commutative field extension of k. (*Hint:* Use Exercises 9.3, 9.4 and that k is a finite field.)
(5) Set $f := [k_D : k]$. Show $f \leq n$.

[1]This exercise sheet will be discussed on January 9. If you have questions or remarks please contact `kay.ruelling@fu-berlin.de` or `kindler@math.fu-berlin.de` or `l.zhang@fu-berlin.de`

(6) Show that $\mathfrak{m} \cdot \mathcal{O}_D = \mathfrak{m}_D^e$.
(7) Show that $n^2 = ef$.
(8) Conclude that $e = n$ and $f = n$.

Exercise 10.3. Let K, D be as in Exercise 10.2.

(1) Show that there exists a subfield $L \subset D$ with $[L : K] = n$ and such that L/K is unramified. (*Hint:* Lift a generator of k_D/k to D.)
(2) Show that

$$\mathrm{Br}(K) = \bigcup_{\substack{L/K \\ \text{fin. unramf. Gal.}}} \mathrm{Br}(L/K) = \mathrm{Br}(K^{\mathrm{ur}}/K),$$

where L/K runs through the finite unramified Galois extensions and K^{ur} is the maximal unramified extension of K.