13. Oktober 2014

Exercise 1 for Number theory III

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Exercise 1.1. Let K be a field and $v : K^\times \to \mathbb{Z}$ a discrete valuation with valuation ring $A = \{a \in K \mid v(a) \geq 0\}$ and maximal ideal $\mathfrak{m} = \{a \in K \mid v(a) > 0\}$. Let c be a real number with $0 < c < 1$ and define a map

$$|-|_{v,c} : K \to \mathbb{R}, \quad x \mapsto |x|_{v,c} = \begin{cases} c^{v(x)} & \text{if } x \neq 0 \\ 0 & \text{else.} \end{cases}$$

(1) Show that $|-|_{v,c}$ is a non-archimedean (or ultrametric) absolute value, i.e. it is multiplicative, has values in $\mathbb{R}_{\geq 0}$, the preimage of $0 \in \mathbb{R}$ is $0 \in K$ and it satisfies the strong triangle equation: $|x+y|_{v,c} \leq \max\{|x|_{v,c}, |y|_{v,c}\}$, for $x, y \in K$.

(2) For $\epsilon > 0$ and $x \in K$ define the ball $B_\epsilon(x) := \{y \in K \mid |x - y|_{v,c} < \epsilon\}$. Say that $U \subset K$ is open if for all $x \in K$ there exists an ϵ such that $B_\epsilon(x) \subset U$. Show this defines a topology on K which coincides with the topology for which a basis of open neighborhoods is given by $x + \mathfrak{m}^n$, $n \geq 0$, $x \in K$. (In particular the topology is independent of c.)

(3) Let $\hat{A} = \varprojlim_n A/\mathfrak{m}^n$ be the completion of A and $\hat{K} = \mathrm{Frac}(\hat{A})$. Let $\hat{v}$ be the discrete valuation on $\hat{K}$ extending v. On $\hat{K}$ we have the non-archimedean absolute value $|-|_{\hat{v},c}$ and it defines a topology as above. Show that the natural inclusion $K \hookrightarrow \hat{K}$ is dense.

(4) Show that any Cauchy sequence in $\hat{K}$ converges. (Recall that a sequence (x_n) in $\hat{K}$ is a Cauchy sequence if for all $\epsilon > 0$ there exists an N such that $|x_m - x_n|_{v,c} < \epsilon$, for all $n, m \geq N$.)

All together we see that $\hat{K}$ is the Cauchy completion of the normed field $(K, |-|_{v,c})$ and it does not depend on the choice of c.

Exercise 1.2. Let A be a ring and $I, J \subset A$ ideals. Assume there exist natural numbers n, m such that $I^m \subset J$ and $J^n \subset I$. Show there is a natural isomorphism

$$\varprojlim_n A/I^n \cong \varprojlim_n A/J^n.$$

Exercise 1.3. Let $p \in \mathbb{Z}$ be a prime number and $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ be the p-adic integers.

(1) Show that the natural inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}_p$ is dense.
(2) Find a sequence (a_n) in $\mathbb{N}$ which converges to $-1 \in \mathbb{Z}_p$.

Exercise 1.4. Let $\mathbb{F}_p[t]$ the polynomial ring in one variable over the field with p elements. Let $f \in \mathbb{F}_p[t]$ be an irreducible polynomial and $\alpha \in \bar{\mathbb{F}}_p$ a root of f in an algebraic closure of $\mathbb{F}_p$. Set $E := \mathbb{F}_p(\alpha) \cong \mathbb{F}_p[t]/(f) \cong \mathbb{F}_q$, where $q = p^{\deg f}$.

(1) Show that $\varphi : \mathbb{F}_p[t] \to E[x]$, $h(t) \mapsto h(x + \alpha)$ is a ring homomorphism and $\varphi^{-1}(x \cdot E[x]) = f \cdot \mathbb{F}_p[t]$.
(2) Show that φ induces an isomorphism $\mathbb{F}_p[t]/(f)^n \to E[x]/x^n$, for all $n \geq 1$.
(3) Conclude that the completion of $\mathbb{F}_p(t)$ at the prime ideal $(f) \subset \mathbb{F}_p[t]$ is isomorphic as complete discrete valuation field to $\mathbb{F}_q((x))$.
(4) Conclude that if K is a global field of characteristic $p>0$ and $\mathfrak{p} \subset \mathcal{O}_K$ is a prime ideal, then the completion $K_\mathfrak{p}$ is a finite field extension of $\mathbb{F}_p((x))$.

Exercise 1.5. Let K be a local field. Show that it is locally compact (i.e. any element $x \in K$ has a compact neighborhood, where compact means: Hausdorff and and any open cover has a finite subcover.)

Hint: Write $K = \mathrm{Frac}(A)$ with A a complete DVR with finite residue field and show that $A/\mathfrak{m}^n$ is finite for all $n \geq 1$.