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Exercise sheet 4
Elliptic Curves [1]

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Exercise 3.1.
(1) Let C be a smooth projective curve of genus 0 over a field and assume that $C(k) \neq \emptyset$. Show that there is an isomorphism of k-schemes $C \cong \mathbb{P}^1_k$. (*Hint:* Take $P \in C(k)$ and use Riemann-Roch to show that there exists a function $f \in k(C)^\times$ which is not algebraic over k and lies in $H^0(C, \mathcal{O}(P))$. Then use Exercise 3.3, (6) to conclude.)
(2) Let $\mathbb{R}$ be the field of real numbers. Show that the scheme $C := \operatorname{Proj} \mathbb{R}[X, Y, Z]/(X^2 + Y^2 + Z^2)$ is a smooth projective curve which is geometrically connected but is not isomorphic to $\mathbb{P}^1_{\mathbb{R}}$.

Exercise 3.2. Let k be an algebraically closed field and E an elliptic curve over k. Choose a point $O \in E(k)$.

(1) Show that the map (of sets) $E(k) \to \operatorname{Pic}^0(E)$, $P \mapsto \mathcal{O}_E([P] - [O])$ is injective. (*Hint:* Else there exists a function $f \in k(E)$ with $\operatorname{div}(f) = [P] - [O]$. Use Exercise 3.3, (6) to get a contradiction.)
(2) Let D be a non-zero divisor of degree 0 on E. Show that there exists a function $f \in k(E)^\times$ such that $\operatorname{div}(f) + D + [O] \geq 0$
(3) Conclude that there exists a point $P \in E$ such that $\operatorname{div}(f) + D + [O] = [P]$.
(4) Show that the map from (1) is bijective.

Exercise 3.3. Let k be a field and $F : (\text{schemes}/k)^{\text{op}} \to (\text{sets})$ be a contravariant functor. Show that the following are equivalent:

(1) There exits a scheme X over k and an isomorphism of functors $\operatorname{Hom}_k(-, X) \to F$ (i.e. X represents F).
(2) There exists an element $\xi \in F(X)$ such that for all S/k and all $a \in F(S)$ there exists a unique k-morphisms $f : S \to X$ with $a = f^*\xi$, where we write $f^* = F(f)$.

[1] This exercise sheet will be discussed on November 11. If you have questions or remarks please contact `kay.ruelling@fu-berlin.de` or `l.zhang@fu-berlin.de`

Exercise 3.4. Let X be a k-scheme. Show that X is a group scheme over k if and only if there exist k-morphisms

$$\mu : X \times_k X \to X, \quad i : X \to X, \quad e : \operatorname{Spec} k \to X$$

such that

$$\mu \circ (\mathrm{id}_X \times i) = e \circ \pi = \mu \circ (i \times \mathrm{id}_X),$$

where $\pi : X \to \operatorname{Spec} k$ is the structure map, and

$$\mu \circ (e \times \mathrm{id}_X) = \mathrm{id}_X = \mu \circ (\mathrm{id}_X \times e), \quad \mu \circ (\mathrm{id}_X \times \mu) = \mu \circ (\mu \times \mathrm{id}_X).$$