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Exercise sheet 3
Elliptic Curves [1]

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Exercise 3.1. Let k be a field and C a smooth projective and geometrically connected curve with function field K. Let $D = \sum_i n_i[x_i]$, $x_i \in C$, be a divisor on C and define a presheaf $\mathcal{O}_C(D)$ on C via

$$C \supset U \mapsto \mathcal{O}_C(D)(U) := \{f \in K^\times \mid \mathrm{div}(f)_{|U} \geq -D_{|U}\},$$

where the restriction maps are induced by the identity map on K. Here we use the following notation: If $E = \sum_j m_j[y_j]$ is a divisor on C, then we set $E_{|U} := \sum_{j \text{ with } y_j \in U} m_j[y_j]$; it is a divisor on U. Show:

(1) $\mathcal{O}_C(D)$ is a sheaf of $\mathcal{O}_C$-modules.
(2) There is an open cover $C = \cup_j U_j$ and functions $f_j \in K^\times$ such that $D_{|U_j} = \mathrm{div}(f_j)_{|U_j}$ and $f_i/f_j \in \mathcal{O}(U_i \cap U_j)^\times$.
(3) Let $\{(U_j, f_j)\}$ be as above. Then $\mathcal{O}_C(D)_{|U_j} = \mathcal{O}_{U_j} \cdot \frac{1}{f_j}$. In particular $\mathcal{O}_C(D)$ is a locally free sheaf of rank 1.
(4) Let 0_C be the zero-divisor. Then $\mathcal{O}_C(0_C) = \mathcal{O}_C$.
(5) Let D' be another divisor on C. Then $\mathcal{O}_C(D) \otimes_{\mathcal{O}_C} \mathcal{O}_C(D') \cong \mathcal{O}_C(D + D')$.
(6) If $D' = D + \mathrm{div}(f)$, for some $f \in K^\times$. Then $\mathcal{O}_C(D') \cong \mathcal{O}_C(D)$.
(7) $\mathcal{H}\mathrm{om}_{\mathcal{O}_C}(\mathcal{O}_C(D), \mathcal{O}_C) \cong \mathcal{O}_C(-D)$.
(8) Assume $D \geq 0$, i.e. D is *effective*, i.e. $n_i \geq 0$ for all i. Set $\underline{D} := \mathrm{Spec}\,(\prod_i \mathcal{O}_{C,x_i}/\mathfrak{m}_i^{n_i})$, where the $\mathfrak{m}_i \subset \mathcal{O}_{C,x_i}$ is the maximal ideal. Then we can define a closed immersion $i : \underline{D} \hookrightarrow C$ such that the following sequence is exact

$$0 \to \mathcal{O}_C(-D) \to \mathcal{O}_C \xrightarrow{i^*} i_*\mathcal{O}_{\underline{D}} \to 0.$$

$\underline{D}$ is called the subscheme associated to D and is often simply denoted by D again.
(9) Assume $\deg(D) := \sum_i n_i[k(x_i) : k] < 0$. Then $\Gamma(C, \mathcal{O}_C(D)) = 0$. (*Hint:* We will prove in the lecture that $\deg(\mathrm{div}(f)) = 0$. You can use it.)

[1] This exercise sheet will be discussed on November 4. If you have questions or remarks please contact `kay.ruelling@fu-berlin.de` or `l.zhang@fu-berlin.de`

Recall: Let X be a noetherian integral scheme with function field K. Denote by $X^{(1)}$ the set of all points $x \in X$ of codimension 1, i.e. the closure $\overline{x}$ of x in X has codimension 1. We assume that for all $x \in X^{(1)}$ the local ring $\mathcal{O}_{X,x}$ is a *DVR* (e.g. X normal or smooth over a field); we denote by $v_x : K^\times \to \mathbb{Z}$ the corresponding normalized discrete valuation. Then by definition

$$\mathrm{CH}^1(X) := \mathrm{coker}(K^\times \xrightarrow{\mathrm{div}} \bigoplus_{x \in X^{(1)}} \mathbb{Z} \cdot \overline{x}),$$

where $\mathrm{div}(f) = \sum_{x \in X^{(1)}} v_x(f) \cdot \overline{x}$ (it is a finite sum as we saw in the lecture).

Exercise 3.2. Let k be a field. Show:

(1) If $X = \mathrm{Spec}\, A$ and A is a unique factorization domain, then $\mathrm{CH}^1(X) = 0$. In particular $\mathrm{CH}^1(\mathbb{A}^n_k) = 0$.
(2) Let $H \subset \mathbb{P}^n_k$ be a hyperplane (i.e. given by the vanishing of a linear homogenous polynomial in $k[x_0, \ldots, x_n]$). Then the map $\mathbb{Z} \to \mathrm{CH}^1(\mathbb{P}^n_k)$, $d \mapsto$ class of $d \cdot H$, is an isomorphism.

Exercise 3.3. Let C be a smooth projective curve over a field k with function field K. Let $f \in K$ be a function.

(1) Show that there is a unique k-morphism $\varphi_f : C \to \mathbb{P}^1_k$ such that on any open affine $U = \mathrm{Spec}\, A \subset C$ on which f is regular (i.e. $f \in A$) the restriction $\varphi_{f|U}$ factors as $U \to \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$, where $U \to \mathbb{A}^1_k$ is induced by $k[t] \to A$, $t \mapsto f$.
(2) Show that the image of φ_f is a point if and only if $f \in K$ is algebraic over k.
(3) Show that φ_f is dominant (i.e. φ_f maps the generic point on C to the generic point on $\mathbb{P}^1_k$) if and only if f is transcendental over k.
(4) Assume f is transcendental over k. Show that φ_f is finite and surjective. (*Hint:* We proved the finiteness in the lecture.)
(5) Assume f is transcendental over k. There are unique effective divisors $\mathrm{div}_+(f)$, $\mathrm{div}_-(f) \geq 0$ on C such that $\mathrm{div}(f) = \mathrm{div}_+(f) - \mathrm{div}_-(f)$. Set $n := \deg(\mathrm{div}_+(f))$. Show that $n \geq 1$ and that the field extension $k(t) = k(\mathbb{P}^1_k) \hookrightarrow K$ induced by φ_f has degree $[K : k(t)] = n$. (*Hint:* By 4 above $\varphi_f^{-1}(\mathbb{A}^1) = \mathrm{Spec}\, B$ with B finite over $k[t]$. Then B is a free $k[t]$-module of rank $= \dim_k B/(f)$.)
(6) Conclude that if there exists a function $f \in K$ with $\deg(\mathrm{div}_+(f)) = 1$, then $\varphi_f : C \to \mathbb{P}^1_k$ is an isomorphism.