November 22, 2013

We make an erratum to the proof of Proposition 2.3. Recall the Proposition:

Proposition 1. *Let X be a smooth connected projective variety defined over a perfect field of characteristic $p > 0$. Then for any stratified bundle $(E_n)_{n\geq 0}$, there is a $n_0 \in \mathbb{N}$ such that the stratified bundle $E(n_0) = (E_{n-n_0})_{n-n_0\geq 0, n\in\mathbb{N}}$ is a successive extension of stratified bundles $(U_n)_{n\in\mathbb{N}}$ with the property that all U_n for $n \in \mathbb{N}$ are μ-stable of slope 0. In particular, all U_n for $n \in \mathbb{N}$ are χ-stable bundles of Hilbert polynomial $p_{\mathcal{O}_X}$.*

Proof. For each $n \geq 0$, we define the socle S_n of E_n (i.e. the maximal polystable subssheaf). The socle is never 0. We write the isotypical decomposition

$$S_n = \oplus_{i\in I_n}\big(V_n(i) \otimes T_n(i)\big)$$

where $V_n(i)$ is stable, not isomorphic to $V_n(j)$ for $i \neq j$, $T_n(i)$ is isomorphic to a direct sum of $\mathcal{O}_X$, I_n is the index set. Let t_n be the maximum rank of the $T_n(i)$ as i varies.

As $F^*V_n(i)$ lies in E_{n-1}, its socle lies in S_{n-1}. So it has the shape $\oplus_{\text{some } j\in I_{n-1}} V_{n-1}(j) \otimes T_{n,n-1}(j)$, where $T_{n,n-1}(j)$ is isomorphic to a direct sum of $\mathcal{O}_X$. So the socle of $F^*\big(V_n(i) \otimes T_n(i)\big)$ has the shape $\oplus_{\text{some } j\in I_{n-1}} V_{n-1}(j) \otimes T_{n,n-1}(j) \otimes T_n(i)$. Thus one has

$$1 \ldots \leq t_n \leq t_{n-1} \leq \ldots \leq t_0.$$

So there is a N such that the sequence $t_n, n \geq N$ is constant, equal to $t \geq 1$ say. We write T for the direct sum of t copies of $\mathcal{O}_X$.

We replace $(E_n)_{n\geq 0}$ by $(E_{n-N})_{n-N\geq 0}$. We write

$$S_n = \oplus_{i\in I(n)}\big(V_n(i) \otimes T\big) \oplus \oplus_j V_n(j) \otimes T_n(j)$$

where $I(n) \subset I_n$ is the subset of indices for which the rank of $T_n(i)$ is equal to t, so rank $T_n(j) < t$. Again the socle of $F^*\big(V_n(i) \otimes T\big)$ has the shape $\oplus_{\text{some } j\in I_{n-1}} V_{n-1}(j) \otimes T_{n,n-1}(j) \otimes T$. As t is maximal, the rank of $T_{n,n-1}(j)$ is 1. Also $V_{n-1}(j)$ can then not be one of the $V_{n-1}(j')$ in the decomposition $\oplus_{\text{some } j'\in I_{n-1}} V_{n-1}(j') \otimes T$ of $F^*\big(V_n(i') \otimes T\big)$ for $i \neq i'$. So the subset of I_{n-1} of indices j appearing in the decomposition of $F^*\big(V_n(i) \otimes T\big)$ lies in $I(n-1)$ and moreover

$$1 \ldots \leq |I(n)| \leq |I(n-1)| \leq \ldots \leq |I(0)|$$

where $|\ |$ denotes the cardinality. So there is a N such that the sequence $|I(n)|, n \geq N$ is constant, equal to I say. So for $n \geq N$, the socle of

$F^*\big(V_n(i)\otimes T\big)$ has to be one of the $V_{n-1}(j)\otimes T$.

We replace $(E_n)_{n\geq 0}$ by $(E_{n-N})_{n-N\geq 0}$ and write

$$W_n = \oplus_{s=1}^I V_n(i_s)\otimes T \subset S_n$$

of rank w_n. So one has

$$1 \leq w_0 \leq \ldots w_n \leq w_{n+1} \ \leq \ldots \leq r.$$

Thus there is a N such that the sequence $w_n, n\geq N$ is constant.

This means that for $n\geq N$, $F^*W_{n+1} = W_n$. We replace $(E_n)_{n\geq 0}$ by $(E_{n-N})_{n-N\geq 0}$. For each isotypical component $V_n(i)$ of W_n, there is a isotypical component $V_{n+1}(j(n,i)))$ of W_{n+1} such that $F^*V_{n+1}(j(n,i)) = V_n(i)$. Thus

$$W(i) := \big(V_0(i), V_1(j(0,i), V_2(j(1,j(0,i)), \ldots\big)$$

is a stratified subsheaf of $(E_n)_{n\geq 0}$, thus a stratified subbundle of $(E_n)_{n\geq 0}$. We define

$$U = (W_n)_{n\geq 0} = (\oplus_{i=1}^I W(i))\otimes T.$$

This is a sum of stratified subbundles, such that all the underlying sheaves are stable bundles.

To finish the proof, we argue by induction on the rank on E/W.

$\square$

Hélène Esnault