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p-CURVATURE of CONNECTIONS, GEOMETRICITY and INTEGRALITY of LOCAL SYSTEMS

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On a X be a smooth complex irreducible variety, one considers complex linear systems V, that is conjugacy classes of irreducible linear representations

$$\pi_1^{\rm top}(X,x)\to GL(r,\mathbb{C})$$

of the topological fundamental group. Equivalently, via the Riemann-Hilbert correspondence [Del70], one considers algebraic regular singular integrable connections (E,∇) associated to V, so in particular $E_{\rm an}^{\nabla_{\rm an}}=V$. Among those are the geometric ones.

From now on, all V's and (E,∇)'s are assumed to be irreducible.

Definition 1. V, equivalently (E,∇), is *geometric* if and only if there is a dense Zariski open subvariety $U\hookrightarrow X$, there is a smooth projective morphism $g:Y\to U$ such that $V|_U$ is a subquotient as a local system of $R^ng_*\mathbb{C}$ for some natural number n.

We call the local system $R^ng_*\mathbb{C}$ a *Gauß-Manin local system.*

Remark 2. Gauß-Manin local systems are defined over $\mathbb{Z}$ thus geometric local systems are defined over the algebraic integers $\bar{\mathbb{Z}}$.

A geometric local system is in particular a variation of Hodge structures. As the category of variations of Hodge structures is semi-simple [Del71], V is geometric if and only if it is a direct factor of a $R^ng_*\mathbb{C}$ as in the definition.

Definition 3. V, equivalently (E,∇), is *finite* if and only it is geometric and g in Definition 1 is finite.

If $V\neq 0$ this implies that $n=0$ and in addition, as $\pi_1^{\rm top}(U,x)\to\pi_1^{\rm top}(X,x)$ is surjective, that one can take g finite as above on $U=X$. A general problem is the following:

Problem 4. Characterize the geometric, resp. finite V (equivalently (E,∇)).

The most general answer is given by the p-curvature conjecture. Let X_S be a model of X over a scheme S of finite type over $\mathbb{Z}$, which we shall assume to be smooth over $\mathbb{Z}$, over which (E,∇) has a model $(E,\nabla)_S$. So in particular E_s is a vector bundle. For any closed point $s\in S$ the p-curvature of $(E,\nabla)_s$ is well defined

$$\psi((E,\nabla)_s):E_s\to F_s^*\Omega^1_{X'_s}\otimes_{\mathcal{O}_{X_s}}E_s.$$

Here $F_s:X_s\to X'_s$ is the relative Frobenius. Then by Cartier descent, one has the following ([Kat70]).

1) $\psi((E,\nabla)_s)$ is nilpotent if and only if there is a filtration Fil on $(E,\nabla)_s$ such that $gr^{\rm Fil}(E,\nabla)_s$ is generated by flat sections.
2) In particular, $\psi((E,\nabla)_s)$ is zero if and only if $(E,\nabla)_s$ is generated by flat sections, or equivalently if and only if there is a coherent vector bundle E'_s on X'_s such that $(E,\nabla)_s=(F_s^*E'_s,\nabla_{\rm can})$ where $\nabla_{\rm can}=1\otimes d$ on $F_s^*E'_s=F_s^{-1}E'_s\otimes_{F^{-1}\mathcal{O}_{X'_s}}\mathcal{O}_{X_s}$.

Conjecture 5 (Grothendieck's p-curvature conjecture)**.** 1) (E,∇) is geometric if and only if its p-curvatures are nilpotent on all closed points s of a dense open $S^0 \hookrightarrow S$.
2) (E,∇) is finite if and only if its p-curvatures are zero on all closed points s of a dense open $S^0 \hookrightarrow S$.

Historically, Grothendieck's p-curvature conjecture was formulated only via 2) and has been discussed and made public by Katz, see [Kat72], [Kat82]. The only complete known result is in rank 1, for which the conjectures 1) and 2) are equivalent and are proven in [Chu85]. Even if we do not touch extensions in this note, let us mention [And04] in which 2) is proven for successive extensions of rank 1 connections, [Kat72] in which 2) is proven for Gauß-Manin connections.

Remark 6. If 1) resp. 2) are true, then if (E,∇) has nilpotent p-curvatures, then $(E,\nabla)_K$, the restriction of (E,∇) to X_K, with $\mathrm{Spec}(K) \to X_S$ a p-adic point, is invariant under Frobenius. Here the Frobenius pull-back of $(E,\nabla)_K$ is well defined, even if the Frobenius does not lift to X_K, as the p-curvature is nilpotent.

Remark 7. If 2) is true, then if (E,∇) has vanishing p-curvatures, then (E_s,∇_s) is not only a connection but is also a D_{X_s}-module.

Theorem 8. *(*[EK18]*) If X is projective, (E,∇) has vanishing p-curvatures, and $(E_s,\nabla)_s$ is a D_{X_s}-module with Tannaka group of order prime to p for infinitely many p's, then (E,∇) is finite.*

Remark 9. The p-curvature conjecture implies in particular that if (E,∇) has nilpotent, resp. vanishing p-curvatures as in Conjecture 5, the monodromy representation ρ^σ defined as the underlying monodromy representation $\rho : \pi_1^{\mathrm{top}}(X,x) \to GL(r,\mathbb{C})$ post-composed by $\sigma^* : GL(r,\mathbb{C}) \to GL(r,\mathbb{C})$ coming from an automorphism σ of $\mathbb{C}$, has associated connection $(E,\nabla)^\sigma$ with nilpotent, resp. vanishing p-curvatures.

We define the following notion.

Definition 10. (E,∇) has *absolute* nilpotent, resp. vanishing p-curvature if and only if for all σ, there is a dense open $S^\sigma \hookrightarrow S$ such that for all closed point $s \in S^\sigma$, $\psi((E,\nabla)_s)$ is nilpotent, resp. vanishing.

So we can subdivide Conjecture 5 as follows.

Conjecture 11. A) (E,∇) has nilpotent, resp. vanishing p-curvatures if and only if it has absolute nilpotent, resp. absolute vanishing p-curvatures.
B) *Absolute p-curvature conjecture*: 1) (E,∇) is geometric if and only if it has absolute nilpotent p-curvatures.
2) (E,∇) is finite if and only if it has absolute vanishing p-curvatures.

Theorem 12 ([EG18])**.** *Let X be smooth projective. Then cohomologically rigid finite determinant (E,∇)'s with absolute vanishing p-curvature are finite.*

Here a *rigid* connection, equivalently a rigid local system, is an isolated point of the de Rham moduli, resp. Betti moduli constructed by Simpson in [Sim94]. It is *cohomologically rigid* if it is a smooth isolated point (i.e. $H^1(X,\mathcal{E}nd^0(\rho)) = H^1(X,\mathcal{E}nd^0(E,\nabla)) = 0$.)

But Simpson conjectures more generally that rigid systems are all geometric.

Conjecture 13 (Simpson's geometricity conjecture)**.** If V has finite determinant and quasi-unipotent monodromies at infinity, and is rigid, then V is geometric.

Here (cohomologically) rigid means that V is an isolated (a smooth isolated) point of the stack of finite type parametrizing such rigid local systems with fixed determinant and given conjugacy classes of monodromy at infinity (see [EG18]).

Remarks 14.
1) Simpson's conjecture implies that if V as in the conjecture is rigid, then the p-curvatures of (E, ∇) are absolutely nilpotent.
2) Simpson's conjecture implies that if V as in the conjecture is rigid, then (E, ∇) restricted to X_K, with $\mathrm{Spec}(K) \to X_S$ a p-adic point is invariant under some power of Frobenius.
3) Simpson's conjecture implies that if V as in the conjecture is rigid, then it is integral.

Theorem 15 ([EG18], [EG18b])**.**
1) *If V has quasi-unipotent monodromies at infinity, has finite determinant, and is cohomologically rigid, then V is integral.*
2) *If X is projective, and V has finite determinant and is rigid, then it has absolute nilpotent p-curvatures, and $(E, \nabla)_K$ is invariant under some power of Frobenius.*

Another way to formulate 2) (without the absolute condition) is to say that $(E, \nabla)_K$ is an isocrystal with Frobenius structure. Using L. Lafforgue's theorem on curves ([Laf02]) together with its translation for overconvergent isocrystals with Frobenius structure ([Abe13]), this implies that if $C \hookrightarrow X$ is a curve such that $\pi_1^{\mathrm{top}}(C, x) \to \pi_1^{\mathrm{top}}(X, x)$ is surjective, then $(E, \nabla)|_{C_K}$ is geometric *as an isocrystal.*

Finally let us mention that on a smooth variety over a finite field, one has ℓ-toℓ' companions as conjectured by Deligne [Del80] and proved by L. Lafforgue [Laf02] and Drinfeld [Dri12], p-to-ℓ companions ([AE16], see also [Ked18]), but we do not know how to construct p-to-p or ℓ-to-p companions, unless X has dimension 1 ([Abe13]). So we mention

Theorem 16 ([EG18b])**.** *Cohomologically rigid connections on X projective, while restricted to X_s, have a full set of companions.*

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