Some Fundamental Groups in Arithmetic Geometry

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Acknowledgements

Thank you to

the organisers for the kind invitation, and much more generally for the friendly and efficient organisation of the whole conference. I can’t speak for the first week, but can for the second one. It was wonderful. Thank you.

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1 Deligne’s conjectures: $\ell$-adic theory

2 Deligne’s conjectures: crystalline theory

3 Malčev-Grothendieck theorem; Gieseker conjecture; de Jong conjecture

4 Relative 0-cycles
Theorem (Deligne ’87)

$X/\mathbb{C}$ smooth connected variety, $r \in \mathbb{N}_{>0}$ given. Then there are finitely many rank $r$ $\mathbb{Q}$-local systems which are direct factors of $\mathbb{Q}$-variations of polarisable pure Hodge structures of a given weight, definable over $\mathbb{Z}$. 

Example (Faltings’ finiteness of abelian schemes on $X$, ’83)

In general, this is a generalisation the version over $\mathbb{C}$ of Faltings' theorem.
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Example (Faltings' finiteness of abelian schemes on \( X \), '83)

In general, this is a generalisation the version over \( \mathbb{C} \) of Faltings' theorem.
Finiteness theorem over $\mathbb{F}_q$

**Theorem (Deligne ’12)**

$X/\mathbb{F}_q$ smooth quasi-projective variety, $r \in \mathbb{N}_{>0}$ given; $D \subset \bar{X}$ an effective Cartier divisor of a normal compactification with support $\bar{X} \setminus X$, and $r \in \mathbb{N}_{>0}$ given. Then there are finitely many irreducible Weil (resp. étale) rank $r$ lisse $\overline{\mathbb{Q}}_\ell$-sheaves with ramification bounded by $D$, up to twist with Weil (resp. étale) characters of $\mathbb{F}_q$. The number does not depend on the choice of $\ell$.

**Corollary (Deligne ’07, Deligne’s conjecture, Weil II, 1.2.10)**

Given an étale lisse $\overline{\mathbb{Q}}_\ell$-sheaf $V$ with finite determinant, the subfield of $\overline{\mathbb{Q}}_\ell$ spanned by the EV of the Frobenii $F_x$ at closed points $x \in |X|$ acting on $V_x$ is a number field.
Theorem over $\mathbb{C}$ is in fact a theorem on $X$ of dimension 1: fixing a good compactification $\tilde{X} \supset X$, with a s.n.c.d. at infinity, then a curve $\tilde{C}$, complete intersection of ample divisors in $\tilde{X}$ in good position, fulfils the Lefschetz theorem

$$\pi_1^{\text{top}}(C := X \cap \tilde{C}) \to \pi_1^{\text{top}}(X).$$
Theorem over $\mathbb{C}$ is in fact a theorem on $X$ of dimension 1: fixing a good compactification $\tilde{X} \supset X$, with a s.n.c.d. at infinity, then a curve $\tilde{C}$, complete intersection of ample divisors in $\tilde{X}$ in good position, fulfils the Lefschetz theorem

$$\pi^\text{top}_1(C := X \cap \tilde{C}) \to \pi^\text{top}_1(X).$$

For $X$ of dimension $\geq 2$ in char. $p > 0$, we do not have a Lefschetz theorem at disposal. So Theorem over $\mathbb{F}_q$ does not reduce to $X$ of dimension 1.
Theorem over $\mathbb{C}$ is in fact a theorem on $X$ of dimension 1: fixing a good compactification $\tilde{X} \supset X$, with a s.n.c.d. at infinity, then a curve $\tilde{C}$, complete intersection of ample divisors in $\tilde{X}$ in good position, fulfills the Lefschetz theorem

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For $X$ of dimension $\geq 2$ in char. $p > 0$, we do not have a Lefschetz theorem at disposal. So Theorem over $\mathbb{F}_q$ does not reduce to $X$ of dimension 1.

Yet one has:
Theorem (Drinfeld '11)

Let $\bar{X} \supset X$ be a projective normal compactification of $X$ smooth over a field $k$, $\Sigma \subset \bar{X}$ be closed of codimension $\geq 2$ such that $(\bar{X} \setminus \Sigma)$ and $(\bar{X} \setminus \Sigma) \cap (\bar{X} \setminus X)$ are smooth, $\bar{C} \subset \bar{X} \setminus \Sigma$ be a smooth projective curve, complete intersection of ample divisors, meeting $\bar{X} \setminus X$ transversally. Then

$$\pi_1^t(C = \bar{C} \cap X) \rightarrow \pi_1^t(X).$$
Tame Lefschetz theorem in char. $p > 0$

**Theorem (Drinfeld ’11)**

Let $\tilde{X} \supset X$ be a projective normal compactification of $X$ smooth over a field $k$, $\Sigma \subset \tilde{X}$ be closed of codimension $\geq 2$ such that $(\tilde{X} \setminus \Sigma)$ and $(\tilde{X} \setminus \Sigma) \cap (\tilde{X} \setminus X)$ are smooth, $\tilde{C} \subset \tilde{X} \setminus \Sigma$ be a smooth projective curve, complete intersection of ample divisors, meeting $\tilde{X} \setminus X$ transversally. Then

$$\pi_1^t(C = \tilde{C} \cap X) \twoheadrightarrow \pi_1^t(X).$$

No need of a good compactification.
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$$\pi^t_1(\mathcal{C} = \bar{C} \cap X) \twoheadrightarrow \pi^t_1(X).$$

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**Proof.**

Bertini to get that restriction to $\mathcal{C}$ of connected finite étale cover of $X$ is connected, tameness and transversality to keep smoothness, thus irreducibility.
If $\tilde{X} \setminus X$ is a s.n.c.d. compactification, Kindler enhances the theorem: if $\tilde{S} \subset \tilde{X}$ is a smooth projective surface, complete intersection of divisors in good position, then

$$\pi_1^t(S = \tilde{S} \cap X) \xrightarrow{\cong} \pi_1^t(X).$$
Theorem (Wiesend ’06, Drinfeld ’11)

Over $X$ quasi-projective smooth over $\mathbb{F}_q$, with $S \subset |X|$ finite:

1) Let $V$ be an irreducible $\overline{\mathbb{Q}}_\ell$-Weil or -étale lisse sheaf, then there is a smooth curve $C \to X$ with $S \subset |C|$, such that $V|_C$ is irreducible;

2) Let $H \subset \pi_1(X)$ be an open normal subgroup, then there is a smooth curve $C \to X$ with $S \subset |C|$, such that $\pi_1(C) \to \pi_1(X)/H$. 

Proof. Uses Hilbert irreducibility ` à la Wiesend.
Wild Lefschetz theorems in char. \( p > 0 \)

**Theorem (Wiesend ’06, Drinfeld ’11)**

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**Proof.**

Uses Hilbert irreducibility à la Wiesend.
Corollary (Drinfeld ’11, Deligne’s conjecture in Weil II, 1.2.10)

1) if \( \det(V) \) is torsion, then \( V \) has weight 0;

2) if \( V \) is an irreducible Weil lisse \( \bar{\mathbb{Q}}_\ell \)-sheaf with determinant of finite order, and \( \sigma \in \Aut(\bar{\mathbb{Q}}_\ell/\mathbb{Q}) \), there is an irreducible Weil lisse \( \bar{\mathbb{Q}}_\ell \)-sheaf \( V_\sigma \), called the \( \sigma \)-companion of \( V \), with determinant of finite order, such that the characteristic polynomials \( f_V, f_{V_\sigma} \in \bar{\mathbb{Q}}_\ell[t] \) of the local Frobenii \( F_x \) satisfy \( f_{V_\sigma} = \sigma(f_V) \).
Corollaries of the wild Lefschetz theorems: weights and companions

Corollary (Drinfeld ’11, Deligne’s conjecture in Weil II, 1.2.10)

1) if $\det(V)$ is torsion, then $V$ has weight 0;
2) if $V$ is an irreducible Weil lisse $\overline{\mathbb{Q}}_\ell$-sheaf with determinant of finite order, and $\sigma \in \text{Aut}(\overline{\mathbb{Q}}_\ell/\mathbb{Q})$, there is an irreducible Weil lisse $\overline{\mathbb{Q}}_\ell$-sheaf $V_\sigma$, called the $\sigma$-companion of $V$, with determinant of finite order, such that the characteristic polynomials $f_V, f_{V_\sigma} \in \overline{\mathbb{Q}}_\ell[t]$ of the local Frobenii $F_x$ satisfy $f_{V_\sigma} = \sigma(f_V)$.

Proof.
Reduce the problem to curves. Then consequence of Lafforgue’s Langlands duality:

1) existence of weights on curves;
2) existence of companions on curves.
Theorem (Kerz-S. Saito ’14)

Let $X$ be a smooth quasi-projective variety over a perfect field $k$, let $X \subset \bar{X}$ be a projective s.n.c.d. compactification, $D$ be an effective divisor with support in $\bar{X} \setminus X$. Define $\pi_1^{ab}(X, D)$ by the condition that a character $\chi : \pi_1(X) \to \mathbb{Q}/\mathbb{Z}$ factors through $\pi_1^{ab}(X, D)$ iff the Artin conductor of $\chi$ pulled-back to any curve $C \to X$ is bounded by the pull-back of $D$ via $\bar{C} \to \bar{X}$. Then Lefschetz holds: for $i : \bar{Y} \subset \bar{X}$ very very ample and in good position w.r.t. $\bar{X} \setminus X$, one has:

$$i_* : \pi_1^{ab}(Y, \bar{Y} \cap D) \to \pi_1^{ab}(X, D)$$

is an isomorphism if $\dim Y \geq 2$, surjective if $\dim Y = 1$. 
Corollary of Abelian Lefschetz theorem: abelian finiteness over \( \mathbb{F}_q \)

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Corollary (Raskind ’92, this formulation by Kerz-S.Saito ’14)

\[ k = \mathbb{F}_q, \text{ then } \ker\left( \pi^\text{ab}_1(X, D) \rightarrow \pi^\text{ab}_1(k) \right) \text{ is finite. (So in particular, this implies Deligne’s finiteness for sums of rank 1 lisse sheaves).} \]
Corollary of Abelian Lefschetz theorem: abelian finiteness over \(\mathbb{F}_q\)

Corollary (Raskind ’92, this formulation by Kerz-S.Saito ’14)

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Proof.

Reduce to curves via de Jong’s alterations (in the more general case \(\tilde{X}\) is a normal compactification) plus the Theorem and apply then CFT.
Right fundamental group with ramification bounded by a $\mathbb{Q}_{\geq 0}$-divisor

Questions

One has the notion of a lisse étale $\mathbb{Q}_\ell$-sheaf $\pi_1(X) \to \text{Aut}(V)$ with ramification bounded by $D$, a positive $\mathbb{Q}$-divisor (Hu-Yang: does not need a good compactification; as for Drinfeld’s Lefschetz theorem for $\pi_1^{t}(X)$). How does one define a quotient $\pi_1(X) \to \pi_1(X, D)$ generalising $\pi_1^{\text{ab}}(X, D)$? Then one could ask for a Lefschetz theorem $\pi_1(C, D_C) \to \pi_1(X, D)$ for a suitable curve $C$ which would reflect Deligne’s finiteness theorem.
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$X$ smooth geometrically connected over a perfect field $k$, $W := W(k)$ the ring of Witt vectors, $K = \text{Frac}(W)$ its field of fractions.
The various categories of isocrystals under consideration

$X$ smooth geometrically connected over a perfect field $k$, $W := W(k)$ the ring of Witt vectors, $K = \text{Frac}(W)$ its field of fractions. One defines the crystalline sites $X/W_n$ as PD-thickenings $(U \hookrightarrow T/W_n, \delta)$, coverings coming from $U \subset X$ Zariski;
The various categories of isocrystals under consideration

$X$ smooth geometrically connected over a perfect field $k$, $W := W(k)$ the ring of Witt vectors, $K = \text{Frac}(W)$ its field of fractions. One defines the crystalline sites $X/W_n$ as PD-thickenings $(U \hookrightarrow T/W_n, \delta)$, coverings coming from $U \subset X$ Zariski; $X/W$ as limit.
• category of \textit{crystals} (i.e. sheaves of $\mathcal{O}_{X/W}$-modules of finite presentation, with transition maps which are isomorphisms) $\text{Crys}(X/W)$, which is $W$-linear;
The various categories of isocrystals under consideration II

- category of crystals (i.e. sheaves of $\mathcal{O}_{X/W}$-modules of finite presentation, with transition maps which are isomorphisms) $\text{Crys}(X/W)$, which is $W$-linear;
- $\mathbb{Q}$-linearisation $\text{Crys}(X/W)_{\mathbb{Q}}$=category of isocrystals, which is $K$-linear, tannakian;
• category of *crystals* (i.e. sheaves of $\mathcal{O}_{X/W}$-modules of finite presentation, with transition maps which are isomorphisms) $\text{Crys}(X/W)$, which is $W$-linear;

• $\mathbb{Q}$-linearisation $\text{Crys}(X/W)_\mathbb{Q}$ = category of *isocrystals*, which is $K$-linear, tannakian;

• absolute Frobenius $F$ acts on $\text{Crys}(X/W)_\mathbb{Q}$;
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- absolute Frobenius $F$ acts on $\text{Crys}(X/W)_{\mathbb{Q}}$;

- largest full subcategory on which every object is $F^\infty$-divisible is $\text{Conv}(X/K) \subset \text{Crys}(X/W)_{\mathbb{Q}}$, the $K$-tannakian subcategory of *convergent* isocrystals (Berthelot-Ogus); (Ogus defines the site of enlargements from $X/W$, then convergent isocrystals are crystals of $\mathcal{O}_{X/K}$-modules of finite presentation).
The various categories of isocrystals under consideration III

- $F : \text{Conv}(X/K) \to \text{Conv}(X/K)$, $(\mathcal{E}, \Phi : F^*\mathcal{E} \xrightarrow{\text{IR}} \mathcal{E}) \mapsto \mathcal{E}$.
The various categories of isocrystals under consideration III

- $F - \text{Conv}(X/K) \to \text{Conv}(X/K)$, $(\mathcal{E}, \Phi : F^*\mathcal{E} \xrightarrow{\text{irr}} \mathcal{E}) \mapsto \mathcal{E}$;
- $F - \text{Conv}(X/K) \mathbb{Q}_p$-linear tannakian;
- $F - \text{Overconv}(X/K)$ fully faithful Kedlaya $\longrightarrow F - \text{Conv}(X/K)$. 

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- $F - \text{Conv}(X/K) \rightarrow \text{Conv}(X/K)$, $(\mathcal{E}, \Phi : F^*\mathcal{E} \rightarrow \mathcal{E}) \mapsto \mathcal{E}$;
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- $F - \text{Overconv}(X/K)$ \xrightarrow{\text{fully faithful Kedlaya}} $F - \text{Conv}(X/K)$;
- $F - \text{Overconv}(X/K)$ consists of those $E$ which have ”unipotent local monodromy” after alteration (Kedlaya);
• Brieskorn over $\mathbb{C}$: Gauß-Manin connections have quasi-unipotent local monodromies;
• Brieskorn over $\mathbb{C}$: Gauß-Manin connections have quasi-unipotent local monodromies;  
so become unipotent after a surjective finite cover of $X$, possibly ramified (Kawamata’s trick);
• Brieskorn over $\mathbb{C}$: Gauß-Manin connections have quasi-unipotent local monodromies; so become unipotent after a surjective finite cover of $X$, possibly ramified (Kawamata’s trick);

• Grothendieck over $\mathbb{F}_q$: lisse $\overline{\mathbb{Q}}_\ell$-sheaves have quasi-unipotent local monodromies (action of local inertia $\mathbb{Z}_\ell(1)$).
• Kedlaya over $k$ (not necessarily perfect): $\mathcal{E} \in F - \text{Overconv}(X/K)$ has 'unipotent monodromy' (in a suitable sense) at infinity after an alteration (uses André-Kedlaya-Mebkhout local result).
• Kedlaya over $k$ (not necessarily perfect): $\mathcal{E} \in F - \text{Overconv}(X/K)$ has 'unipotent monodromy' (in a suitable sense) at infinity after an alteration (uses André-Kedlaya-Mebkhout local result).
There are blow-ups at infinity: analog to resolution of turning points over $\mathbb{C}$ (Kedlaya/T. Mochizuki).
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There are blow-ups at infinity: analog to resolution of turning points over $\mathbb{C}$ (Kedlaya/T. Mochizuki).

- From the definition:

On $X$ proper, $F - \text{Overconv}(X/K) = F - \text{Conv}(X/K)$. 
Over $\mathbb{F}_q$, $q = p^s$, define $F_{\mathbb{F}_q} = F^s - \text{Overconv}(X/K)$, so $K = \text{Frac}W(\mathbb{F}_q)$-linear; *abuse of notations* $F - \text{Overconv}(X/K)$. 
• Over $\mathbb{F}_q$, $q = p^s$, define $F_{\mathbb{F}_q} = F^s - \text{Overconv}(X/K)$, so $K = \text{Frac}W(\mathbb{F}_q)$-linear; abuse of notations $F - \text{Overconv}(X/K)$.

• $L$-linearisation, for $K \subset L \subset \bar{\mathbb{Q}}_p$, $L \to \bar{\mathbb{Q}}_p$, defines the category $F - \text{Overconv}(X/K)_{\bar{\mathbb{Q}}_p}$. 
Irreducible objects in $F − \text{Overconv}(X/K) \bar{\mathbb{Q}}_p$ with finite determinant;
Irreducible objects in $F - \text{Overconv}(X/K)_{\overline{\mathbb{Q}}_p}$ with finite determinant; are analog to irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaves with finite determinant;
Irreducible objects in $F - \text{overconv}(X/K)_{\overline{\mathbb{Q}}_p}$ with finite determinant; are analog to irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaves with finite determinant; Upon bounding ramification at infinity (correct notion for $F - \text{overconv}(X/K)_{\overline{\mathbb{Q}}_p}$?),
Irreducible objects in $F - \text{Overconv}(X/K)\overline{\mathbb{Q}}_p$ with finite determinant; are analog to irreducible lisse $\overline{\mathbb{Q}}_\ell$-sheaves with finite determinant; Upon bounding ramification at infinity (correct notion for $F - \text{Overconv}(X/K)\overline{\mathbb{Q}}_p$?), are analog over $\mathbb{C}$ to irreducible $\mathbb{Q}$-variations of polarisable pure Hodge structures of pure weight definable over $\mathbb{Z}$. 
Crystalline version ("petits camarades cristallins") on curves

Theorem (Abe, Crystalline version of Lafforgue’s theorem ’13)

Let $X$ be a smooth curve over $\mathbb{F}_q$. Then

1) an irreducible overconvergent $\overline{\mathbb{Q}}_p$-$F$-isocrystal with finite determinant is $\iota$-pure of weight 0;

2) an irreducible lisse $\overline{\mathbb{Q}}_\ell$-étale sheaf with finite determinant has an overconvergent $\overline{\mathbb{Q}}_p$-$F$–isocrystal companion and vice-versa.
Crystalline version (petits camarades cristallins) on higher dimensional varieties

No wild Lefschetz theorem for $F$-overconvergent isocrystals

In particular

1) one does not know whether irreducible $F$-isocrystals with finite determinant are $\iota$-pure of weight 0 on $X$, smooth quasi-projective variety over $\mathbb{F}_q$;

2) a fortiori, one does not have a number field capturing the EV of $F$-isocrystals acting on the stalks at closed points $\text{Spec} \mathbb{F}_q(x)$;

3) nor does one have a crystalline version of Deligne's finiteness theorem.

So: no higher dimensional generalisation of Drinfeld/Deligne.
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3) nor does one have a crystalline version of Deligne’s finiteness theorem.

So: no higher dimensional generalisation of Drinfeld/Deligne.
Abe’s Čebotarev theorem

Theorem (Abe ’13)

On $X$ quasi-projective smooth over $\mathbb{F}_q$, $\iota$-pure (or mixed, $\iota : \overline{\mathbb{Q}}_p \cong \mathbb{C}$) semi-simple objects in $F – \text{Overconv}(X/K)$ are determined by their local EV at closed points.
1. Deligne’s conjectures: $\ell$-adic theory

2. Deligne’s conjectures: crystalline theory

3. Malčev-Grothendieck theorem; Gieseker conjecture; de Jong conjecture

4. Relative 0-cycles
Rather than considering analogies between some irreducible complex local systems (‘motivic’ ones) with some lisse $\bar{\mathbb{Q}}_{\ell}$-sheaves (irreducible with finite determinant) over $\mathbb{F}_q$, and with some overconvergent $\bar{\mathbb{Q}}_p$-$F$ isocrystals (irreducible with finite determinant), one can raise the...
Weaker analogies

Rather than considering analogies between some irreducible complex local systems (’motivic’ ones) with some lisse $\overline{\mathbb{Q}}_\ell$-sheaves (irreducible with finite determinant) over $\mathbb{F}_q$, and with some overconvergent $\overline{\mathbb{Q}}_p$- $F$ isocrystals (irreducible with finite determinant), one can raise the

**Question**

what is the analog of complex local systems on $X$ over $\mathbb{C}$ for $X$ over a perfect field of characteristic $p > 0$?
Infinitesimal site

\[ X \text{ smooth over a char. 0 field } k; \]
Infinitesimal site and crystals in characteristic 0

Infinitesimal site

$X$ smooth over a char. 0 field $k$; $X_\infty: U \hookrightarrow T$ infinitesimal thickening of a Zariski open $U$; coverings from the $U$s.

$\{\text{finitely presented crystals on } X_\infty\} = \{(E, \nabla)\}$, $E$ coherent sheaf and $\nabla$ flat connection (thus $E$ is locally free); $k$-linear category (assume here $k$ = field of constants of $X$, i.e. $X$ geometrically connected over $k$).
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$k$-linear category (assume here $k = \text{field of constants of } X$, i.e. $X$ geometrically connected over $k$);
### Theorem (Malčev ’40-Grothendieck ’70)

If \( X \) is smooth over \( \mathbb{C} \); then \( \pi_{\text{ét}}^1(X) = \{1\} \) implies there are no non-trivial crystals in the infinitesimal site (with regular singularities at infinity in case \( X \) is not projective).

### Proof.

Use Riemann-Hilbert correspondence to translate to finite dimensional complex local systems.
Malčev-Grothendieck’s theorem

Theorem (Malčev ’40-Grothendieck ’70)

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Proof.

Use Riemann-Hilbert correspondence to translate to finite dimensional complex local systems.

Then $\pi_1^{\text{top}}(X(\mathbb{C}))$ is an abstract group of finite type, so $\rho : \pi_1^{\text{top}}(X(\mathbb{C})) \to GL(r, \mathbb{C})$ factors through $\rho_A : \pi_1^{\text{top}}(X(\mathbb{C})) \to GL(r, A)$, $A/\mathbb{Z}$ of finite type, and $\rho = 1$ iff $\rho_A = 1$ iff $\rho_a : \pi_1^{\text{top}}(X(\mathbb{C})) \to GL(r, \kappa(a)) \forall$ closed point $a \in \text{Spec}(A)$.
So \{ \text{finite étale category} \} trivial implies \{ \text{infinitesimal crystals} \} (regular singular) trivial.
Conservativity

So \{\text{finite étale category}\} trivial implies \{\text{infinitesimal crystals}\} (regular singular) trivial.

More modest question: analogs in char. $p > 0$ of this conservativity theorem? (Terminology ‘conservativity’ borrowed from Ayoub’s work).
Infinitesimal site

As in char. 0: $X$ smooth over a char. $p > 0$ field $k$; $X_\infty: U \hookrightarrow T$ infinitesimal thickening of a Zariski open $U$; coverings from the $U$s.
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\{\text{finitely presented crystals on } X_\infty\} = \{\mathcal{O}_X - \text{coherent } \mathcal{D}_X - \text{modules}\}
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Infinitesimal site and crystals in characteristic $p > 0$

**Infinitesimal site**

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$$\{\text{finitely presented crystals on } X_\infty\} = \{\mathcal{O}_X - \text{coherent } D_X - \text{modules}\} = \{F - \text{d.c.s.}\} \text{ (divided coherent sheaves) (Cartier isomorphism, Katz’ theorem).}$$
Infinitesimal site

As in char. 0: $X$ smooth over a char. $p > 0$ field $k$; $X_\infty$: $U \hookrightarrow T$ infinitesimal thickening of a Zariski open $U$; coverings from the $U$s.

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$k$-linear category (assume here $k = \text{field of constants of } X$, i.e. $X$ geometrically connected over $k$).
Gieseker’s conjecture

Gieseker’s conjecture ’75

On $X$ projective smooth over $k = \bar{k}$ of char. $p > 0$, $\pi^\text{ét}_1(X) = \{1\}$ implies that there are no non-trivial crystals in the infinitesimal site.
Gieseker’s conjecture ’75

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Theorem (E-Mehta ’10)

Conjecture has a positive answer.
Variants of Gieseker’s conjecture

- $X$ not proper: theory of *regular singular* crystals in the infinitesimal site developed by Kindler (’13), so that for those with finite monodromy, it coincides with the notion of *tame* quotient of $\pi_1^{\text{ét}}(X)$.
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- Yet no ramification theory, so far.
• $X$ not proper: theory of \textit{regular singular} crystals in the infinitesimal site developed by Kindler ('13), so that for those with finite monodromy, it coincides with the notion of \textit{tame} quotient of $\pi_1^{\text{ét}}(X)$.
• Yet no ramification theory, so far.
• So far no extension of the conservativity theorem, except for the tame abelian quotient (Kindler '13) and for $X = \text{smooth locus of a normal projective variety and } k = \overline{\mathbb{F}}_q$ (E-Srinivas '14, using an improvement of Grothendieck’s LEF theorem by Bost, '14).
On the proof of Gieseker’s conjecture

- $E \in F - \text{d.c.s.}$ implies $E \in \text{Coh}(X)$ is locally free. In fact, this is a Tannakian category over $k$. 
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- $E \in F - d.c.s.$ implies $E \in \text{Coh}(X)$ is locally free. In fact, this is a Tannakian category over $k$.
- $E \in F - d.c.s.$ implies $0 = c_{i,\text{crys}}(E) \in H^{2i}_{\text{crys}}(X/W), \ i \geq 1$. 

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Fundamental Groups
Utah, 27-29-30 July 2015
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On the proof of Gieseker’s conjecture

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- Thus they have moduli points in Langer’s moduli of semi-stable pure sheaves with trivial numerical Chern classes.
- Hrushovsky’s theorem then guarantees the existence of a Frobenius invariant vector bundle on a specialization of $X$ over $\overline{F}_p$, which yields a non-trivial finite étale cover of this one.
The categories considered were already presented. Again we assume $X$ projective smooth geometrically connected over $k$ perfect of char. $p > 0$. We consider

- $k = \mathbb{F}_q$, $\mathbb{F}$-convergent isocrystals $\mathbb{F}^{-\text{Conv}}(X/K) = \mathbb{F}^{-\text{Conv}}(X/K)$;
- $k$ perfect, convergent isocrystals and isocrystals $\mathbb{C} \mathbb{y}r \mathbb{s}(X/K) \hookrightarrow \mathbb{C} \mathbb{r} \mathbb{s}(X/W)$.
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- $k = \mathbb{F}_q$, $F$-convergent isocrystals
  
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  $\text{Conv}(X/K) \hookrightarrow \text{Crys}(X/W)_{\mathbb{Q}}$. 
On $X$ projective smooth over $k$ perfect of char. $p > 0$, $\pi_1^{\text{ét}}(X \otimes_k \bar{k}) = \{1\}$ implies that there are no non-trivial isocrystals.
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Theorem (E-Shiho, ’15)

Partial positive answer.
de Jong’s conjecture

On $X$ projective smooth over $k$ perfect of char. $p > 0$, $\pi_1^{\text{ét}}(X \otimes_k \bar{k}) = \{1\}$ implies that there are no non-trivial isocrystals.

Theorem (E-Shiho, ’15)

*Partial positive answer.*

In the sequel, we report on it, raising a few questions on the way.
\( \pi_1^{\text{ét}}(X \otimes_k \bar{k}) = \{1\} \) implies \( H^1(X, \mathcal{O}_{X/W}^\times) = 0 \), thus rank 1 locally free crystals, and thus isocrystals, are trivial, as well as \( H^1(X/W) = 0 \), thus successive extensions of trivial rank 1 crystals, thus isocrystals, are trivial.
Abelian case

\[ \pi_1^{\text{ét}}(X \otimes_k \bar{k}) = \{1\} \text{ implies } H^1(X, O_X^\times) = 0, \text{ thus rank 1 locally free crystals, and thus isocrystals, are trivial,} \]

as well as \( H^1(X/W) = 0 \), thus successive extensions of trivial rank 1 crystals, thus isocrystals, are trivial.

Thus the conjecture essentially predicts a relation between the "non-abelian" part of \( \pi_1^{\text{ét}}(X) \) and irreducible isocrystals of higher rank.
\( \pi_1^{\text{ét}}(X \otimes_k \bar{k}) = \{1\} \) implies \( H^1(X, \mathcal{O}_{X/W}^\times) = 0 \), thus rank 1 locally free crystals, and thus isocrystals, are trivial, as well as \( H^1(X/W) = 0 \), thus successive extensions of trivial rank 1 crystals, thus isocrystals, are trivial.

Thus the conjecture essentially predicts a relation between the "non-abelian" part of \( \pi_1^{\text{ét}}(X) \) and irreducible isocrystals of higher rank.

At least when \( p \geq 3 \) it is so; for \( p = 2 \) those statements are less direct and follow from the whole proof.
Gauß-Manin $F$-convergent isocrystal

**Theorem (E-Shiho ’15)**

Let $f : Y \to X$ be a smooth projective morphism over $X$ smooth projective over $k$ perfect. If $\pi_1^{\text{ét}}(X \otimes_k \bar{k}) = \{1\}$, then the $F$-convergent isocrystal $R^i f_*$ is trivial in $\text{Conv}(X/K)$.
Let $f : Y \to X$ be a smooth projective morphism over $X$ smooth projective over $k$ perfect. If $\pi_1^{\text{ét}}(X \otimes_k \overline{k}) = \{1\}$, then the $F$-convergent isocrystal $R^i f_*$ is trivial in $\text{Conv}(X/K)$.

Model of Proof

Assume $f$ was an abelian scheme and $k = \mathbb{F}_q$. May assume $X$ has a rational point $x_0$. Then (argument of Faltings): $\pi_1^{\text{ét}}(X)$ acts on $R^i f_* \mathbb{Q}_\ell$ via $\text{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_q)$, thus by the Honda-Tate theorem, all geometric fibres of $f$ are isogeneous. Thus for all closed points $x \in |X|$, $H^1(Y_x/\text{Frac} W(k(x))) = H^1(Y_{x_0}/K) \otimes_K \text{Frac} W(k(x))$, thus the isocrystal $R^i f_*$ is trivial in $\text{Conv}(X/K)$.
Proof.

Over $k = \mathbb{F}_q$, one replaces the motivic argument (Honda-Tate) by Abe’s Čebotarev’s density theorem.
Proof.

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Yields triviality of the semi-simplification of $R^if_*$ in $\text{Conv}(X/K)$ after specialisation over $\mathbb{F}_q$. 

Thus $R^if_* \in \text{Conv}(X/K) \subset \text{Crys}(X/W_{\mathbb{Q}})$ is trivial, as we already saw that there are no non-trivial extensions.

From now on, we discuss the general case.
Proof.

Over $k = \mathbb{F}_q$, one replaces the motivic argument (Honda-Tate) by Abe’s Čebotarev’s density theorem.

Yields triviality of the semi-simplification of $R^i f_*$ in $\text{Conv}(X/K)$ after specialisation over $\mathbb{F}_q$.

Apply base change to get it over $k$. Yields triviality of the semi-simplification of $R^i f_*$ in $\text{Conv}(X/K)$ over $k$.

Thus $R^i f_* \in \text{Conv}(X/K) \subset \text{Crys}(X/W)_\mathbb{Q}$ is trivial, as we already saw that there are no non-trivial extensions.

From now on, we discuss the general case.
Lemma

$X$ smooth over $k$ perfect, $\mathcal{E} \in \text{Crys}(X/W)_\mathbb{Q}$, there is a $p$-torsion-free $E \in \text{Crys}(X/W)$ with $E_{\mathbb{Q}} = \mathcal{E}$, called a lattice.

Proof.

Given any $E \in \text{Crys}(X/W)$, with $E_{\mathbb{Q}} = \mathcal{E}$, the surjective maps $E/Ker(p^n + 1) \rightarrow E/Ker(p^n)$ stabilise, as one sees locally on finitely many open affines $U$, as then $\text{Crys}(U/W) \cong \text{MIC}(\hat{U}W/W)_{qn}$, the quasi-nilpotent flat connections on a formal lift.
Existence of lattices

**Lemma**

\( X \) smooth over \( k \) perfect, \( \mathcal{E} \in \text{Crys}(X/W)_\mathbb{Q} \), there is a \( p \)-torsion-free \( E \in \text{Crys}(X/W) \) with \( E_\mathbb{Q} = \mathcal{E} \), called a lattice.

**Proof.**

Given any \( E \in \text{Crys}(X/W) \), with \( E_\mathbb{Q} = \mathcal{E} \), the surjective maps \( E/\text{Ker}(p^{n+1}) \to E/\text{Ker}(p^n) \) stabilise, as one sees locally on finitely many open affines \( U \), as then \( \text{Crys}(U/W) \cong \text{MIC}(\hat{U}_W/W)^{qn} \), the quasi-nilpotent flat connections on a formal lift.
A sheaf $\mathcal{E}$ is said to be *locally free* if it has a locally free lattice $E$, so $E_{\mathbb{Q}} = \mathcal{E}$, that is equivalently if $E_X$, the value of $E$ on $X \hookrightarrow X$, viewed in $\text{Coh}(X)$, is locally free.

**Question:** Are all $E \in \text{Crys}(X/\mathbb{W})$ locally free? A positive answer would ease the understanding of de Jong's conjecture.
Locally free lattices

\( \mathcal{E} \) is said to be \textit{locally free} if it has a locally free lattice \( E \), so \( E_\mathbb{Q} = \mathcal{E} \), that is equivalently if \( E_X \), the value of \( E \) on \( X \hookrightarrow X \), viewed in \( \text{Coh}(X) \), is locally free.

Question

Are all \( \mathcal{E} \in \text{Crys}(X/W)_\mathbb{Q} \) locally free?

A positive answer would ease the understanding of de Jong’s conjecture.
Theorem (E-Shiho ’15)

Let $E \in \text{Crys}(X/W)$ be a lattice.

1) If $E$ is locally free, then $0 = c_{i,\text{crys}}(E_X) \in H^{2i}(X/W)$, $i \geq 1$.

2) If $E_\mathbb{Q} \in \text{Conv}(X/K)$, then $0 = c_{i,\text{crys}}(E_X) \in H^{2i}(X/K)$, $i \geq 1$. 
Proof of the vanishing of the crystalline Chern classes of the value on $X$ of lattices

**Proof.**

On 1): there are (at least) **two ways**;
Proof of the vanishing of the crystalline Chern classes of the value on $X$ of lattices

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*Modified splitting principle:* on $X \subset D \to \mathbb{P} W$ PD-hull, one considers the quotient $\Omega_D^\bullet \to \bar{\Omega}_D^\bullet$ of DGAs defined by $dx[n] = x^{[n-1]}dx$. This defines the quotient $\Omega_{\mathbb{P}(E_D)}^\bullet \to \bar{\Omega}_{\mathbb{P}(E_D)}^\bullet$ of DGAs by moding out by the 'same' kernel, where $E_D$ is the value of $E$ on $X \hookrightarrow D$. Let $\pi : \mathbb{P}(E_D) \to D$ be the principal bundle.
Proof of the vanishing of the crystalline Chern classes of the value on $X$ of lattices

Proof.

On 1): there are (at least) two ways;

*Modified splitting principle:* on $X \subset D \to \mathbb{P}_W$ PD-hull, one considers the quotient $\Omega^\bullet_D \to \bar{\Omega}^\bullet_D$ of DGAs defined by $dx[n] = x^{[n-1]}dx$. This defines the quotient $\Omega^\bullet_{\mathbb{P}(E_D)} \to \bar{\Omega}^\bullet_{\mathbb{P}(E_D)}$ of DGAs by moding out by the ‘same’ kernel, where $E_D$ is the value of $E$ on $X \hookrightarrow D$. Let $\pi : \mathbb{P}(E_D) \to D$ be the principal bundle. One shows $\nabla$ on $E_D$ defines a splitting $\bar{\Omega}^\bullet_D \to R\pi^*\bar{\Omega}^\bullet_{\mathbb{P}(E_D)} \xrightarrow{\tau} \bar{\Omega}^\bullet_D$, defining a partial connection on $\pi^*E_D$ with value in $\pi^*\bar{\Omega}^1_D$, which is shown to respect $\mathcal{O}_{\mathbb{P}(E_D)}(1)$. \qed
Proof of the vanishing of the crystalline Chern classes of the value on $X$ of lattices II

Proof.

*Equating $\mathcal{O}$ and $\overline{dR}$-cohomology*: for $X$ smooth, define $D_\bullet$ to be the simplicial scheme defined by $D_n = $PD-hull of the diagonal in $\mathbb{P}^{x(n+1)}_W$. Then $H^i(X/W) = H^i_{dR}(D_\bullet)(:= H^i(D_\bullet, \Omega^\bullet_{D_\bullet})) = H^i_{Zar}(D_\bullet, \mathcal{O})$. 
Proof.

*Equating $\mathcal{O}$ and $\overline{dR}$-cohomology:* for $X$ smooth, define $D_\bullet$ to be the simplicial scheme defined by $D_n = \text{PD-hull of the diagonal in } \mathbb{P}^n_W$. Then $H^i(X/W) = H^i_{dR}(D_\bullet)(:= H^i(D_\bullet, \Omega^\bullet_{D_\bullet})) = H^i_{\text{Zar}}(D_\bullet, \mathcal{O})$. For $E$ given, $E_X$ defines $e : X_\bullet \to BGL(r)$, $BGL(r)$ simplicial scheme, $X_\bullet$ coming from a Čech simplicial scheme associated to local trivialisations of $E_X$.
Proof.

Equating $\mathcal{O}$ and $dR$-cohomology: for $X$ smooth, define $D_\bullet$ to be the simplicial scheme defined by $D_n = \text{PD-hull of the diagonal in } \mathbb{P}^{(n+1)}_W$. Then $H^i(X/W) = H^i_{dR}(D_\bullet) := H^i(D_\bullet, \Omega^\bullet_{D_\bullet}) = H^i_{\text{Zar}}(D_\bullet, \mathcal{O})$. For $E$ given, $E_X$ defines $e: X_\bullet \to BGL(r)$, $BGL(r)$ simplicial scheme, $X_\bullet$ coming from a Čech simplicial scheme associated to local trivialisations of $E_X$. Do a simplicial version $D_\bullet$ of the $D_\bullet$ above. Then $E$ has a value $E_{D_\bullet}$. So $e^*: H^{2i}(BGL(r)/W) \to H^{2i}(X/W)$ factors through $H^{2i}(D_\bullet/W) = H^{2n}(D_\bullet, \mathcal{O})$, thus $e$ factors through $H^{2i}(BGL(r)/W) \to H^{2i}(BGL(r), \mathcal{O})$, which is trivial for $i \geq 1$. \qed
Proof.

2) Show, the class of $E_X$ in $K_0(X)$, where $E$ is a lattice of $E \in \text{Crys}(X/W)_{\mathbb{Q}}$, depends only on $E$. Thus since $E \in \text{Conv}(X/K)$ is $F^\infty$-divisible, $\text{ch}_{i,\text{crys}}(E_X) \in H^{2i}(X/K)$ is $p^{i\infty}$-divisible, thus $= 0$, thus $0 = c_{i,\text{crys}}(E_X) \in H^{2i}(X/K)$. 
Lift of trivial crystals over $k$

Lemma

Assume $E \in \text{Crys}(X/W)$ is a lattice, such that $\exists m \in \mathbb{N} \geq 0$ such that $(F^m)^*E_X \in \text{MIC}(X/k)^{\text{q}}$ is trivial. Then if $\pi_1^{\text{ét, ab}}(X \otimes_k \overline{k}) = \{1\}$, $E \in \text{Crys}(X/W)$ is trivial.

Begin of Proof.

$F^* : \text{Crys}(X/W)_\mathbb{Q} \to \text{Crys}(X/W)_\mathbb{Q}$ is fully faithful, so may assume $E_X$ trivial.
Proof of lift II

Proof.

For $D$ PD-hull of $X \subset \mathbb{P}_W$, with $D_n = D \otimes_W W_n$, has

$$\text{Ker} \left( \text{MIC}(D_{n+m}) \to \text{MIC}(D_n) \right) \cong M(r \times r, H^1_{dR}(D_m)) \ 1 \leq \forall \ m \leq n. \quad (*)$$
Proof of lift II

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For $D$ PD-hull of $X \subset \mathbb{P}_W$, with $D_n = D \otimes_W W_n$, has

$$\text{Ker}(\text{MIC}(D_{n+m}) \to \text{MIC}(D_n)) \cong M(r \times r, \text{H}^1_{dR}(D_m)) \ 1 \leq m \leq n. \quad (*)$$

$(\pi^\text{ét,ab})_1(X \otimes_k \bar{k}) = \{1\}$ implies $(F^a)^* \text{H}^1(X/k) = (F^a)^* \text{H}^1_{dR}(D_1) = 0 \ \exists a \in \mathbb{N}$ and that such that $\text{H}^1_{dR}(D_n) \to \text{H}^1_{dR}(D_1) \ \forall n \geq N \in \mathbb{N}_{>0}.$
Proof.

For $D$ PD-hull of $X \subset \mathbb{P}_W$, with $D_n = D \otimes W W_n$, has

$$\text{Ker}(\text{MIC}(D_{n+m}) \to \text{MIC}(D_n)) \cong M(r \times r, H^1_{dR}(D_m)) 1 \leq \forall m \leq n. \quad (\star)$$

$\pi_{1,\text{ét},\text{ab}}(X \otimes_{\bar{k}} k) = \{1\}$ implies $(F^a)^* H^1(X/k) = (F^a)^* H^1_{dR}(D_1) = 0 \ \exists a \in \mathbb{N}$ and that such that $H^1_{dR}(D_n) \xrightarrow{0} H^1_{dR}(D_1) \ \forall n \geq \exists N \in \mathbb{N}_{>0}.$

Applying $(\star)$ to $(n, m) = (1, 1), (2, 1), \ldots, (N - 1, 1)$, we conclude that there is $b \in \mathbb{N}$ depending only on $X$ such that $((F^b)^* E)_{D_N}$ is trivial. Replace $E$ by $(F^b)^* E$, may assume $E_{D_N}$ is trivial.

Applying $(\star)$ to $(n, m) = (2N, N)$ we conclude $E_{D_{N+1}} = \text{image } E_{D_{2N}}$ via $M(r \times r, H^1(D_N)) \to M(r \times r, H^1(D_1))$, is trivial.
Proof.

For $D$ PD-hull of $X \subset \mathbb{P}_W$, with $D_n = D \otimes_W W_n$, has

$$\text{Ker} \left( \text{MIC}(D_{n+m}) \to \text{MIC}(D_n) \right) \cong M(r \times r, H^1_{dR}(D_m)) \quad 1 \leq \forall \ m \leq n.$$  \hfill (\star)

$\pi^\text{ét,ab}_1 (X \otimes_k \bar{k}) = \{1\}$ implies $(F^a)^* H^1(X/k) = (F^a)^* H^1_{dR}(D_1) = 0$ $\exists a \in \mathbb{N}$ and that such that $H^1_{dR}(D_n) \to H^1_{dR}(D_1)$ $\forall n \geq 3 \ N \in \mathbb{N}_{>0}$.

Applying (\star) to $(n, m) = (1, 1), (2, 1), \ldots, (N - 1, 1)$, we conclude that there is $b \in \mathbb{N}$ depending only on $X$ such that $((F^b)^* E)_{D_N}$ is trivial. Replace $E$ by $(F^b)^* E$, may assume $E_{D_N}$ is trivial.

Applying (\star) to $(n, m) = (2N, N)$ we conclude $E_{D_{N+1}} = \text{image } E_{D_{2N}}$ via $M(r \times r, H^1(D_N)) \to M(r \times r, H^1(D_1))$, is trivial.

One continues, etc.
Theorem (E-Shiho ’15)

Let $X$ be smooth projective over $k = \bar{k}$ of char. $p > 0$. Let $\mathcal{E}$ be $\in \text{Conv}(X/K)$ or be locally free in $\text{Crys}(X/W)_{\mathbb{Q}}$. If $\pi^\text{ét}_1(X) = \{1\}$, $\mu_{\text{max}}(\Omega^1_X) < N(r)$ for a certain positive number $N(r)$ discussed below, and the irreducible constituents of the Jordan-Hölder filtration of $\mathcal{E}$ have rank $\leq r$, then $\mathcal{E}$ is trivial.

$N(1) = \infty$, so complete theorem for extension of rank 1 isocrystals;
Theorem (E-Shiho ’15)

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$N(1) = \infty$, so complete theorem for extension of rank 1 isocrystals; $N(2) = 2$, $N(3) = 1$, $N(r) = 1/M(r)$, $M(r) = \max \text{lcm}(a, b)$, $a, b \geq 1$, $a + b \leq r$.

Restriction on $E$-in $\text{Conv}(X/K)$ or locally free- comes from the fact that the vanishing of the crystalline Chern classes of $E_X$ is proven under this assumption.
Trivializing a crystal

Theorem (E-Shiho ’15)

Let $X$ be smooth projective over $k = \bar{k}$ of char. $p > 0$. Let $\mathcal{E}$ be $\in \text{Conv}(X/K)$ or be locally free in $\text{Crys}(X/W)_{\mathbb{Q}}$. If $\pi^\text{ét}_1(X) = \{1\}$, $\mu_{\text{max}}(\Omega^1_X) < N(r)$ for a certain positive number $N(r)$ discussed below, and the irreducible constituents of the Jordan-Hölder filtration of $\mathcal{E}$ have rank $\leq r$, then $\mathcal{E}$ is trivial.

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Restriction on $\mathcal{E}$-in $\text{Conv}(X/K)$ or locally free- comes from the fact that the vanishing of the crystalline Chern classes of $E_X$ is proven under this assumption. The rôle of the stability assumptions will become clear in the proof.
Proof.

- As in the proof of the triviality of the crystals on the infinitesimal site, one has to bound the problem.
Proof of the Theorem on trivialisation of a crystal

Proof.

- As in the proof of the triviality of the crystals on the infinitesimal site, one has to bound the problem.

- One proves a *Langton type theorem*: starting with $\mathcal{E}$ irreducible, there is a lattice $E$ such that $E_X \in \text{MIC}(X/k)$ is semi-stable, so yields a point of Langer’s moduli $M$ of semi-stable connections on pure sheaves with vanishing Chern classes.
Proof of the Theorem on trivialisation of a crystal

Proof.

• As in the proof of the triviality of the crystals on the infinitesimal site, one has to bound the problem.

• One proves a *Langton type theorem*: starting with $\mathcal{E}$ irreducible, there is a lattice $E$ such that $E_\mathcal{X} \in MIC(X/k)$ is semi-stable, so yields a point of Langer’s moduli $M$ of semi-stable connections on pure sheaves with vanishing Chern classes.

• Using vanishing of crystals in the infinitesimal site (Theorem E-Mehta), and quasi-projectivity of $M$, one concludes that if $(F^a)^*E_\mathcal{X}$ was semi-stable for a large enough but finite, then $(F^a)^*E_\mathcal{X}$ would be trivial. So the lifting lemma would finish the proof.
Proof of the Theorem on trivialisation of a crystal

Proof.

- As in the proof of the triviality of the crystals on the infinitesimal site, one has to bound the problem.
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- Using vanishing of crystals in the infinitesimal site (Theorem E-Mehta), and quasi-projectivity of $M$, one concludes that if $(F^a)^*E_X$ was semi-stable for a large enough but finite, then $(F^a)^*E_X$ would be trivial. So the lifting lemma would finish the proof.
- The slope assumption enables one to assume semi-stability of $(F^a)^*E_X$ for a certain $a \geq 0$. 

Hélène Esnault, Freie Universität Berlin
Fundamental Groups
Utah, 27-29-30 July 2015 48 / 60
Base change for étale cohomology for torsion coefficients of order prime to $p$

SGA 4,5, IV Thm.1.2.: Let $A$ be an henselian discrete valuation ring (d.v.r.), with residue field $k$ of characteristic $p > 0$. Let $X/A$ be a scheme, $(n, p) = 1$. Then if $X/A$ is proper, one has base change, that is the restriction homomorphism $H^i_\text{ét}(X, \mathbb{Z}/n) \xrightarrow{\text{rest}} H^i_\text{ét}(Y, \mathbb{Z}/n)$ is an isomorphism, where $Y = X \otimes_A k$. 

Question: What is a motivic version of the base change theorem?

Theorem (Kerz-E-Wittenberg '15)

Answer for relative 0-cycles.
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Question

What is a motivic version of the base change theorem?

Theorem (Kerz-E-Wittenberg ’15)

Answer for relative 0-cycles.

In the sequel, we report in it, raising a few questions on the way.
Why relative 0-cycles?

Examples

• Let $X/A$ be a K3-surface, with $k = \bar{k}$ and $A$ large enough so $NS(X_{\bar{K}})$ is defined over $K = \text{Frac}(A)$. Then $NS(X_K) \to NS(Y)$ is an injection of torsion-free lattices of possibly different (Néron-Severi) ranks, e.g. assume $Y$ is supersingular! Thus composite

$$\text{Pic}(X)/n \xrightarrow{\text{rest. surj.}} \text{Pic}(X_K)/n \xrightarrow{\text{sp}} \text{Pic}(Y)/n,$$

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- Rosenschon-Srinivas ’07 produced examples of 3-dimensional $X_K$, $K$ a $p$-adic field with $CH^2(X_K)/n$ infinite, thus $CH^2(X)/n$ infinite for a regular model $X/A$ as $CH^2(X)/n \xrightarrow{\text{rest. surj.}} CH^2(X_K)/n$; yet $CH^2(Y)/n$ is finite thus restriction $CH^2(X)/n \to CH^2(Y)/n$ can’t be injective.
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• So restriction neither surjective nor injective.
Half way: from ’motivic’ cohomology on $X$ to étale cohomology on $Y$

So if we keep this direct (naïve ?) formulation, we should exclude from the study relative cycles of positive dimension.
Half way: from ’motivic’ cohomology on $X$ to étale cohomology on $Y$

So if we keep this direct (naïve ?) formulation, we should exclude from the study relative cycles of positive dimension.

For relative 0-cycles one has

**Theorem (Sato-S.Saito ’10)**

Assume $A$ excellent, henselian discrete valuation ring, with finite or separably closed residue field $k$ of characteristic $p > 0$. Assume $X/A$ projective, irreducible strict normal crossings (s.n.c.) scheme (so $X$ in particular is regular) of relative dimension $d$. Then the cycle map $c_X : CH_1(X)/n \to H^{2d}_{\text{ét}}(X, \mathbb{Z}/n(d))$ is an isomorphism.
Sato-Saito’s theorem deals with the \textit{cycle map} $c_X$. We want to lift this information to an information of the following kind, possibly enlarging the range of applicability for more general $A$ and $k$:
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\begin{align*}
\text{CH}_1(X)/n & \xrightarrow{\rho} C(Y)/n \\
\downarrow c_X & \quad \quad \quad \downarrow c_Y \\
H^{2d}_{\text{ét}}(X, \mathbb{Z}/n(d)) & \xrightarrow{\text{rest}} H_{\text{ét}}^{2d}(Y, \mathbb{Z}/n(d)).
\end{align*}
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(⋆)
There are a priori two ways to think:

1) We keep the cycle group $\text{CH}^1(X)/n$ and have to define a cycle group $\text{C}(Y)/n$ and show it is an isomorphism;

2) We replace $\text{CH}^1(X)/n$ by motivic cohomology $H^{2d}_{\text{mot}}(X, Z/n(d))$ and define $\text{C}(Y)/n$ to be motivic cohomology $H^{2d}_{\text{mot}}(Y, Z/n(d))$; then one hopes functoriality defines $\rho$ and one hopes that one can show it is an isomorphism.

On the other hand, one conjectures $\text{CH}^1(X) = H^{2d}_{\text{mot}}(X, Z(d))$ for the $(A, k)$ we shall consider.

In fact we mix the two viewpoints: we want to construct a restriction homomorphism $\rho: \text{CH}^1(X)/n \to H^{2d}_{\text{mot}}(Y, Z/n(d))/n$, which is then an isomorphism and lifts the base change isomorphism.
Formulation of the problem II

There are a priori two ways to think:

1) We keep the cycle group $CH_1(X)/n$ and have to define a cycle group $C(Y)/n$ and the restriction $\rho$ and show it is an isomorphism;

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In fact we mix the two viewpoints: we want to construct a restriction homomorphism \( \rho : CH_1(X)/n \to H^{2d}_{mot}(Y, \mathbb{Z}(d))/n \), which is then an isomorphism and lifts the base change isomorphism.
Recall

\[ CH_1(X) / n \xrightarrow{\rho \downarrow} C(Y) / n \]

\[ c_X \downarrow \quad \quad \quad c_Y \downarrow \]

\[ H^{2d}_{\text{ét}}(X, \mathbb{Z} / n(d)) \xrightarrow{\text{rest}} H^{2d}_{\text{ét}}(Y, \mathbb{Z} / n(d)). \]
Theorem

Theorem (Kerz-E-Wittenberg ’15)

1) Let $Y$ be a strict normal crossings variety of dimension $d$ defined over a perfect field $k$. Then there is a description of $H_{\text{mot}}^{2d}(Y, \mathbb{Z}[^1_p](d))$ as a quotient of $\mathbb{Z}[Y^{\text{sm}}]$ by explicit relations, and $H_{\text{mot}}^{2d}(Y, \mathbb{Z}[^1_p](d)) = H_{\text{Nis}}^d(Y, K_d^M)[^1_p]$.

2) Assume $A$ excellent henselian d.v.r., with perfect char. $p > 0$ residue field, and $X/A$ be a projective s.n.c. scheme. Then the following holds.

   i) If $A$ has equal char. then $\text{CH}_1(X)/n = H_{\text{Nis}}^d(X, K_d^M/n)$ (Kerz’ theorem), $\rho$ is then defined via restriction on $K_d^M$ and one has ($\star$);

   ii) If $k$ is finite or algebraically closed, one has ($\star$);

   iii) If $((d-1)!, n) = 1$, in particular if $d = 2$, one has ($\star$).
Proof.

Ad 1): uses localisation in motivic cohomology on $Y$, then duality to relate $H^{2d}_{c,\text{mot}}(Y^{\text{sm}}, \mathbb{Z}/n(d))$ with Suslin homology $= \mathbb{Z}[Y^{\text{sm}}]/(\mathcal{R}I)$, $\mathcal{R}I$ spanned by certain $(C, g)$, $C \subset Y$ integral 1-dimensional subscheme not contained in $Y^{\text{sing}}$, $g$ rational function which is a unit generically and equal to 1 along $Y^{\text{sing}}$.
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Result: $H^2_{mot}(Y, \mathbb{Z}/n(d)) = \mathbb{Z}[Y^{sm}]/(\mathcal{R}I, \mathcal{R}II)$, with $\mathcal{R}II$ spanned by $(C, g)$, $C$ simple n.c. curve and $g$ unit along $Y^{sing}$. 

\qed
Proof.

Ad 2): \( \rho \) uniquely defined by writing \( CH_1(X) \) as a quotient of \( Z^g_1(X) \subset Z_1(X) \) spanned by \( A \)-flat 1-cycles which intersect \( Y \) in \( Y^{sm} \).

i) A equal char.: one uses Kerz’ theorem showing 
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CH_1(X)/n = H^d(X_{\text{Nis}}, \mathcal{K}_{X,d})
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to define \( \rho \);

ii) \( k \) finite or algebraically closed: one uses \'etale cohomology and the Kato conjecture proven by Kerz-S.Saito to define \( \rho \);

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In all cases, once \( \rho \) is defined, one uses geometry to show it is an isomorphism.
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In all cases, once $\rho$ is defined, one uses geometry to show it is an isomorphism.
Some consequences

Corollary

1) If $k$ is finite, then $\text{CH}_0(X_K)/n$ is finite (already a consequence of Sato-S.Saito);

2) $A = k[[t]]$, $k$ $p$-adic field, then $\text{CH}_0(X_K)/n$ is finite.

Proof.

Ad 1): This is the link to the first lectures: Class Field Theory plus the Kato conjecture enable one to show finiteness of $\mathcal{C}(Y)/n$. One then applies the theorem.
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Ad 2): One uses again the Kato conjecture (and a result of Forré).
Higher relative dimension?

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Questions

- What would be a (conjectural) formulation for cycles of higher relative dimension?
- For $K$ a $p$-adic field, for which motivic cohomology groups of $X_K$ does one have finiteness, and for those for which one does not have finiteness, does one have meaningful quotients which are finite?
Higher relative dimension?

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• What would be a (conjectural) formulation for cycles of higher relative dimension?

• For $K$ a $p$-adic field, for which motivic cohomology groups of $X_K$ does one have finiteness, and for those for which one does not have finiteness, does one have meaningful quotients which are finite?

• What about mod $p$, and what about replacing $Y$ by its thickenings $Y_m$?

Assuming Gersten conjecture for Milnor $K$-theory on $X$ one has a restriction homomorphism $CH_1(X)/n \to \lim_{m} H^d_{Nis}(Y_m, \mathcal{K}_d^M/n)$ (possibly $p$ divides $n$) and one could ask, when $A$ is the ring of integers of a number field, whether the prosystem is constant. This is related to Colliot-Thélène’s conjecture on the structure $CH_0(X_K)$, which should be of the shape $\mathbb{Z}$ (for the degree) + a finite group + a free lattice over $\mathbb{Z}_p$ + a divisible group.