PERIOD MAPPINGS, FROM COMPLEX TO P-ADIC

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Tempered coverings and fundamental group(s)

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1. Analytic continuation
2. p-adic geometry
3. Topological and étale coverings
4. Tempered coverings and fundamental group.
5. Examples and final picture.

1 Analytic continuation

Let S be a topological space, $\mathcal{F}$ an abelian sheaf on S.

For $f \in \Gamma(\mathcal{U}, \mathcal{F})$, $\operatorname{supp}(f) := \{u \in \mathcal{U} \mid f_u \neq 0\}$

Def. We say $\mathcal{F}$ satisfies the principle of unique continuation if $\forall\, \mathcal{U} \subseteq S$ open, $f \in \Gamma(\mathcal{U}, \mathcal{F})$, then $\operatorname{Supp}(f) \subseteq \mathcal{U}$ is open.

Lemma. $\mathcal{F}$ satisfies pple of unique cont.

$$\Downarrow$$

$\forall\, f, g \in \Gamma(\mathcal{U}, \mathcal{F})$, $\mathcal{U} \subseteq S$ op. conn., then $f = g \iff f_s = g_s$ for some $s \in \mathcal{U}$.

Moreover, if S is locally connected, $\Uparrow$ also holds

Def. We say $\mathcal{F}$ satisfies the monodromy principle if

- let $\Gamma : [a, b] \times [0, 1] \to S$ continuous, $\Gamma|_{\{a\} \times [0,1]} = \{x_a\}$

  $\Gamma|_{\{b\} \times [0,1]} = \{x_b\}$

Let $f_{x_a} \in \mathcal{F}_{x_a}$. Assume that $\forall t \in [0,1]$, f_{x_a} extends to a global section f^{Γ_t} of $\Gamma_t^* \mathcal{F}$ on $[a,b]$.
Then, this extension is unique and $f^{\Gamma_t}(b) \in \mathcal{F}_{x_b}$ is independent of t.

Prop. $\mathcal{F}$ satisfies pple unique cont.

$$\Downarrow$$

$\mathcal{F}$ satisfies pple of monodromy.

Moreover, if S is locally arcwise connected, $\Uparrow$ also holds.

Ex: 1) S complex manifold, then $\mathcal{O}_S$ satisfies pple of unique continuation.

2) $\mathcal{F}$ locally constant ab. sheaf on any top. sp. S satisfies pple of unique cont.

Q. What do we have in the p-adic picture?

(2) p-adic geometry

There are several ways to do analysis over $\mathbb{C}_p$.
Note that $\mathbb{C}_p$ is totally disconnected w.r.t. the p-adic topology, so there is no obvious way to do it.

For example, how can we make analysis on $\mathbb{Q}$?

- 1st approach: restrict the amount of open sets that you consider in the topology. For instance, if you take as $\underbrace{\ }_{\text{(a basis of)}}$ admissible opens the balls $B(\underset{\mathbb{Q}}{q_0}, \underset{\mathbb{Q}_{>0}}{r}) = \{q \in \mathbb{Q} \mid |q - q_0| < r\}$

  then ~~locally constant~~ we can make analysis.

· For example, locally constant functions are constant.

· 2^{nd} approach: add points to $\mathbb{Q}$ (the irrational pts) and get a nicer top sp ——— $\mathbb{R}$

So, what do we do in $\mathbb{C}_p$?

· 1^{st} approach: work on affinoid algebras (is like giving the analytic functions) and admissible opens. ⇝ Rigid geom. à la Tate

· 2^{nd} approach: add points to get a nicer top. space ⇝ Berkovich geom.

$\mathbb{C}_p$ already complete, (alg. closed) How to add points?

Berkovich geometry

· Let $(K, |\cdot|)$ be a complete field, A an affinoid K-algebra,

(i.e. a $\underbrace{K\text{-alg}}_{\text{Banach}}$. s.t. $\exists$ surjection $K\{r^{-1}t\} \longrightarrow A$

$|\cdot|$ non-arch. — means $(A, |\cdot|_A)$ A complete wrt

$$K\{r^{-1}t\} = \left\{ \sum_{i=0}^{\infty} a_i t^i \;\middle|\; |a_i| r^i \xrightarrow[i\to\infty]{} 0 \right\})$$

Then the (Berkovich) $\widehat{\text{spectrum}}$ is defined as follows $\widehat{\mathcal{M}(A)}$

a) As a set: $\mathcal{M}(A) := \{$ bounded multiplicative real seminorms of $A\}$

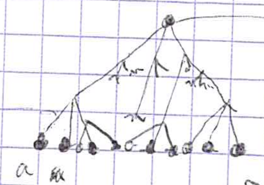
b) Topology: weakest s.t. the maps $\mathcal{M}(A) \to \mathbb{R}_{\geq 0}$, $\chi \longmapsto \chi(f)$ continuous for any $f \in A$.

c) Sheaf of rings $\mathcal{O}_{\mathcal{M}(A)}$ ...

We build analytic spaces by glueing so called affinoid domains, which are a nice class of affinoid spaces.
This glueing is tricky, (the building blocks are compact!) we give no details here.
It gives an analytification functor.

Drawings. Assume $K = \mathbb{C}_p$.

1) $D(0,1^+) \rightsquigarrow$ ~~$\mathcal{M}(D(0,1^+))$~~
$\mathcal{M}(\mathbb{C}_p\{t\})$



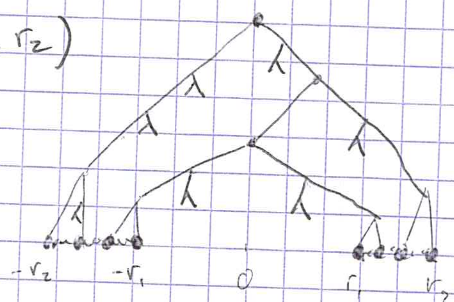
$|\cdot|_{D(0,1^+)}$ = supremum norm

type I pts, together are like usual p-adic $D(0,1^+)$. They correspond to the seminorms $|f|_a := |f(a)|_{\mathbb{R}}$

$|\cdot|_{D(a,r^+)}$

2) $\mathbb{A}^1_{\mathbb{C}_p} \rightsquigarrow$ glue all the balls $\mathcal{M}(\mathbb{C}_p\{r^{-1}t\})$, $r \geq 0$.

3) Annuli $A(0, r_1, r_2)$



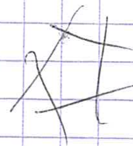
4) Curves (elliptic) (wlog, take $E/\mathbb{C}_p$)

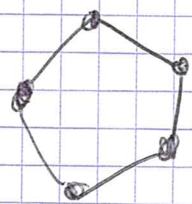
Step 1: choose a semistable model

Step 2: draw the skeleton of the analytification, which is the dual intersection complex of the special fiber

Good reduction

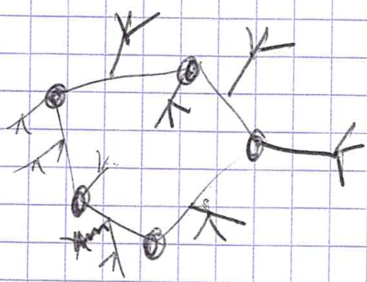
Bad reduction





Step 3: add the rest

$y^2 = x^3 + x + 1$



$y^2 = x^3 + px + p$

Similar with other curves.

From now on, let $K \subset \mathbb{C}_p$ complete.

Q. Are these ~~spaces~~ Berkovich spaces nice?

A. Quite nice!

- Locally compact
- Locally countable at ∞
- Locally arcwise connected (!)

We have the analytification functor

$$\left\{ \begin{array}{l} \text{separated schemes locally} \\ \text{of fin. type } / K \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{paracompact (strictly)} \\ \text{analytic } K\text{-spaces} \end{array} \right\}$$

Paracompact: every open cover has an open refinement which is locally finite.

Locally finite: every open intersects just finitely many others.

Def (K-manifold) Let S a paracomp. strict K-an. sp. It is a <u>K-manifold</u> if

- any $s \in S$ has a nbhd $U(s)$ isomorphic to an affinoid subdomain of some sp. V_s which admits locally an étale morphism to the affine space $\mathbb{A}^{\dim S}$

If $K = \mathbb{C}_p$, we call it p-adic manifold.

Fact If X is an alg. K-variety, then X smooth $\Leftrightarrow$ X^{an} K-manifold.

Rem. We can define a notion of étale morphism. If f is alg., then f étale $\Leftrightarrow$ f^{an} étale

Thm (Berkovich) K-manifolds are (locally arcwise connected and) locally simply connected !!!

Prop The structure sheaf of any p-adic manifold S satisfies the pple of unique continuation (and hence the pple of monodromy)

③ Topological and étale coverings

- Because of the theorem, we can construct topologically a universal covering $(\tilde{S}, \tilde{s})$ of our K-manifold (S, s) such that $\tilde{S}/\pi_1^{top}(S,s) \simeq S$

Complex setting $\rightsquigarrow$ notions of topological coverings and étale coverings coincide.

In p-adic setting $\rightsquigarrow$ NO!! We have:
- Too few topological coverings
- Too many étale coverings

~~Topolog~~ Let S be a connected K-manifold

Def. The morphism $f: S' \longrightarrow S$ is a

① covering
② étale covering
③ topological covering

if S is covered by open subsets $\mathcal{U}$ s.t. $\bigsqcup V_j = f^{-1}\mathcal{U} \longrightarrow \mathcal{U}$

and the restrictions $f|_{V_j}$ are
① finite
② étale finite
③ isomorphisms

Rem. Today, we are interested in ② $\rightarrow$ $\mathrm{Cov}_S^{\text{ét}}$ and ③ $\rightarrow$ Cov_S^{top}.

Moreover, the subclass of finite étale coverings (i.e. we take finitely many V_j) will be denoted Cov_S^{alg}

$$\begin{tikzcd} & \mathrm{Cov}_S^{\text{ét}} & \\ \mathrm{Cov}_S^{alg} \arrow[ur] & & \mathrm{Cov}_S^{top} \arrow[ul, hook] \end{tikzcd}$$

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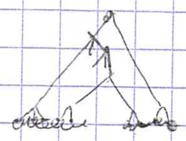
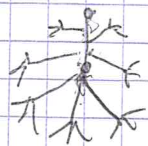
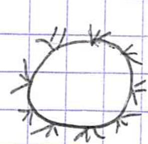
Examples

① Topological coverings:

- Annulus  is simply connected.

- ~~(Analitifica~~ $\mathbb{P}^1_{\mathbb{C}_p}$ is also simply conected 
  $\mathbb{A}^1_{\mathbb{C}_p}$

- Moreover, analytifications of smooth proj. curves with good reduction are also contractible!

- Non trivial top. covering: Tate curve  has.

→ In general, too few.

② Étale coverings

i)

- (dJ95): $\mathbb{A}^1_{\mathbb{C}_p}$ has non-trivial étale coverings:

$$\log : D_{\mathbb{C}_p}(1, 1^-) \longrightarrow \mathbb{A}^1_{\mathbb{C}_p} \; ; \; 1+z \longmapsto z - \frac{z^2}{2} + \frac{z^3}{3} \cdots$$

Indeed, if we cover $\mathbb{A}^1_{\mathbb{C}_p}$ with the $\overset{\text{open}}{\smile}$ discs

$$D_m := D\left(0, \left(|p|^{-m+\frac{1}{p-1}}\right)^-\right)$$

we have that

$$\log^{-1}(D_m) = \bigsqcup_{\zeta \in \mu_{p^\infty}} \underbrace{\left\{ q \in D_{\mathbb{C}_p}(1,1^-) \,\middle|\, \zeta q^{p^m} \in D\left(1, \left(|p|^{\frac{1}{p-1}}\right)^-\right) \right\}}_{\text{finite étale } / D_m}$$

Moreover,

$$\pi_1^{\text{ét}}(\mathbb{A}^1_{\mathbb{C}_p}, 0) \twoheadrightarrow \bigcup_n \mu_{p^n}(\mathbb{C}_p) \simeq \mathbb{Q}_p/\mathbb{Z}_p$$

and for $\mathbb{Q}_p$, we have

$$\pi_1^{\text{ét}}(\mathbb{A}^1_{\mathbb{Q}_p}, 0) \twoheadrightarrow \bigcup_n \mu_{p^n}(\overline{\mathbb{Q}}_p) \rtimes \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p).$$

Rem: the logarithm is an example of a rigid analytic period map for p-divisible groups. (c.f. end of Salim's talk)

2) $\pi_1^{\text{ét}}(\mathbb{P}^{h-1}_{\mathbb{C}_p}) \twoheadrightarrow SL_h(\mathbb{Q}_p)$

3) Lots of period mappings give étale coverings.

4) Arbitrary top. coverings of a finite étale covering are also ~~an~~ étale coverings.

Lemma. Let $S' \to S$ be an étale covering, $R \subset S' \times_S S'$ a union of conn. comp which is an equiv. relation on S' over S. Then S'/R (viewed as an ét. sheaf quotient) is representable by an étale covering $S'' \to S$.

Def. In this situation, we say $S'' \to S$ is a quotient étale covering of $S' \to S$.

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④ Tempered covering and fundamental group

Idea: we want an intermediate category inside $Cov_S^{ét}$ that ~~includes Co~~ contains Cov_S^{top} and Cov_S^{alg}

Def. Let $f \in Cov_S^{ét}$. We say that is temperate if $\exists$ quotient of

$$\begin{tikzcd} & T' \arrow[dl, "top"'] \arrow[dr, dashed] & \\ T \arrow[dr, "alg"'] & & S' \arrow[dl, "f"] \\ & S & \end{tikzcd}$$

Note $T \in Cov_S^{alg}$, $T' \in Cov_S^{top}$, $S' \in Cov_S^{ét}$.

We denote this category as Cov_S^{temp}

$Cov_S^{ét} \supset Cov_S^{temp} \supset Cov_S^{alg}$, $Cov_S^{temp} \supset Cov_S^{top}$

Fundamental groups

Let $Cov_S^{\bullet}$ be a full subcat. of $Cov_S^{ét}$ s.t.

* stable under connected components
* " fiber products / S
* " quotients.

Examples: $Cov_S^{ét}$, Cov_S^{top}, Cov_S^{alg}, Cov_S^{temp}

Let ~~For~~ $\bar{s} \to S$ geom. pt, and let

$$F_{\bar{s}}^{ét} : Cov_S^{ét} \to Sets : S' \mapsto \{\text{geom. pts } \bar{s}' \in S' \text{ above } \bar{s}\},$$

and let $F_{\bar{s}}^{\bullet} := F_{\bar{s}}^{ét}\Big|_{Cov_S^{\bullet}}$

Def. $\pi_1^{\bullet}(S, \bar{s}) := \mathrm{Aut}\, F_{\bar{s}}^{\bullet}$

Rem: If S is the analytif. of an algebraic smooth K-variety S^{alg} (K still of char $=0$), then
$$\pi_1^{alg}(S^{alg}, \bar{s}) = \pi_1^{alg}(S, \bar{s}).$$

Facts:

1) $\pi_1^{\circ}(S, \bar{s})$ is a separated pro-discrete (hence tot. disc.) top. space.
2) $\pi_1^{\circ}(S, \bar{s})$ doesn't depend on $\bar{s}_m$ (if S conn.)
3) $F_{\bar{s}}^{\circ} : Cov_S^{\circ} \longrightarrow \pi_1^{\circ}(S, \bar{s})$-Sets, and

Thm. Fully faithful, and extends to an equiv. of cat.
$$\left\{ \begin{array}{c} \text{disjoint unions of obj} \\ \text{of } Cov_S^{\circ} \end{array} \right\} \longrightarrow \pi_1^{\circ}(S, \bar{s})\text{-Sets}$$

&

connected components $\longleftrightarrow$ $\pi_1^{\circ}(S, \bar{s})$-orbits

Galois $\longleftrightarrow$ $\pi_1^{\circ}(S, \bar{s}) \to G$ surj. cont.

Def (Galois covering) A connected étale covering $S' \to S$ is Galois if $S = S'/Aut_S S'$

Cor. $\pi_1^{\circ}(S, \bar{s})$ is a pro-discrete gp $\Longleftrightarrow$ In Cov_S°, any connected covering is dominated by a Galois covering.

Example: 1) $\pi_1^{et}(\mathbb{P}^1, \bar{s})$ is not a pro-discrete gp.

Indeed, $\pi_1^{et}(\mathbb{P}^1, \bar{s}) \twoheadrightarrow SL_2(\mathbb{Q}_p) \longrightarrow SL_2(\mathbb{Q}_p)/SL_2(\mathbb{Z}_p)$

The preimage of $SL_2(\mathbb{Q}_p)/SL_2(\mathbb{Z}_p)$ can't be open normal subgps in

because the only normal subgp of $SL_2(\mathbb{Q}_p)$ contained in

$SL_2(\mathbb{Z}_p)$ is $\{\pm \mathrm{id}\}$.

2) $\pi_1^{temp}(S,\bar{s})$ is always a pro-discrete gp.

~~Let's compute some tempered fundamental groups.~~

Relation between fundamental gps

white board

1) $Cov_S^{top} \hookrightarrow Cov_S^{temp}$ induces

$$\pi_1^{temp}(S,\bar{s}) \longrightarrow \pi_1^{top}(S,s)$$

which is a surjective hom.

2) $Cov_S^{alg} \hookrightarrow Cov_S^{temp}$ induces

$$\pi_1^{temp}(S,\bar{s}) \longrightarrow \pi_1^{alg}(S,\bar{s})$$

with dense image.

Hence $\left(\pi_1^{temp}(S,\bar{s})\right)^{\wedge} = \pi_1^{alg}(S,\bar{s})$. (profin. compl.)

In particular, $\pi_1^{temp}(S,\bar{s})$ is not too small.

Moreover, $\pi_1^{temp}(S,\bar{s})$ is not too big:

Prop. If $\dim S = 1$, $\pi_1^{temp}(S,\bar{s}) \hookrightarrow \pi_1^{alg}(S,\bar{s})$ injective!

π_1^{temp} encapsulates info of bad reduction of

Ex: 1) in particular,

$$\log: D_{\mathbb{C}_p}(1,1^-) \to \mathbb{A}^1_{\mathbb{C}_p}$$

is not temperate, since $\pi_1^{alg}(\mathbb{A}^1_{\mathbb{C}_p}, 0) = 0 \Rightarrow$

$\Rightarrow \pi_1^{temp}(\mathbb{A}^1_{\mathbb{C}_p}, 0) = 0$

2) Similarly, $\pi_1^{temp}(\mathbb{P}^1_{\mathbb{C}_p}, \bar{s}) = 0$

3) $\mathbb{G}_m = \mathbb{P}^1 \setminus \{0,\infty\}$

We know $\pi_1^{top}(\mathbb{G}_m, 1) = 0$

Moreover, connected fin. étale coverings are Kummer ($\mathbb{G}_m \to \mathbb{G}_m$, $x \mapsto x^n$), so

$$\pi_1^{temp}(\mathbb{G}_m, 1) = \pi_1^{alg}(\mathbb{G}_m, 1) \cong \hat{\mathbb{Z}}$$

Similarly, $\pi_1^{temp}(\mathbb{G}_m^n, 1) \simeq \hat{\mathbb{Z}}^n$

4) $\mathbb{P}^1 \setminus \{0, 1, \infty\}$: $\pi_1^{temp}(\mathbb{P}^1 \setminus \{0,1,\infty\}, \bar{s})$ is very difficult!

We have: $\pi_1^{top}(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \bar{s}) = 0$

$$\pi_1^{alg}(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \bar{s}) = \hat{\mathbb{Z}} * \hat{\mathbb{Z}}$$

$$\hookuparrow$$

$$\pi_1^{temp}(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \bar{s})$$

Fact: it is not locally compact! Maybe sth more in next talks...

5) Elliptic curves

a) Good reduction $\rightsquigarrow$ $\pi_1^{temp}(S, 1) \simeq \hat{\mathbb{Z}} \cdot \hat{\mathbb{Z}}$

b) Bad reduction $\rightsquigarrow$ $\pi_1^{temp}(S, 1) \simeq \hat{\mathbb{Z}} \cdot \mathbb{Z}$

6) Abelian varieties:

Let A of $\dim(A) = g$ and $\underbrace{\dim(\text{toric part of semistable reduction})}_{\text{rank}(\pi_1^{top}(A, \bar{e}))} = d$

Then, $\pi_1^{temp}(A, \bar{e}) \simeq \hat{\mathbb{Z}}^{2g-d} \cdot \mathbb{Z}^d$

7) Let C be a curve of genus ≥ 2, A its Jacobian variety.

$$\left(\pi_1^{temp}(C, \bar{p})\right)^{ab} \simeq \pi_1^{temp}(A, \bar{o})$$

<u>Some more facts about π_1^{temp}</u>

Let $K = \overline{K}$ c.v.f. of char 0.

<u>Facts</u> 1) Let $X \to Y$ birational map between <u>smooth and proper</u> K-schemes. Then

$$\pi_1^{temp}(X^{an}, \bar{x}) \simeq \pi_1^{temp}(Y^{an}, \bar{y})$$

2) If $\underset{\rightsquigarrow \text{ isometry}}{\Omega \supset \overline{K}}$ alg. closed ~~[illegible]~~ c.v.f.,

$$\pi_1^{top}(X_{K'}, x') \simeq \pi_1^{top}(X, x)$$

$$\pi_1^{temp}(X_{K'}, x') \simeq \pi_1^{temp}(X, x)$$

3) $$\pi_1^{top}(X \times Y, (x, y)) \simeq \pi_1^{top}(X, x) \times \pi_1^{top}(Y, y)$$

$$\pi_1^{temp}(\qquad) \simeq \pi_1^{temp}$$

info of bad reduction of fin. ét. coverings

info of bad reduction of S

$$\pi_1^{temp}(S, \bar{s}) \twoheadrightarrow \pi_1^{top}(S, \bar{s})$$

$$\widehat{\pi_1^{temp}(S, \bar{s})} = \pi_1^{alg}(S, \bar{s})$$

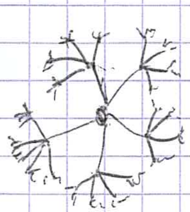
$\dim S = 1 \rightsquigarrow \pi_1^{temp}(S, \bar{s}) \hookrightarrow \pi_1^{alg}(S, \bar{s})$

Auxiliar 1

(1). Relation between different fundamental groups on Berkovich spaces

(2). Berkovich analytification of an elliptic curve:

with good reduction



with bad reduction