Lecture 2

Ramification groups

Ex: $\begin{matrix} C \ni x \\ \downarrow \\ \mathbb{F}_p \end{matrix} \rightsquigarrow k(C)_x :=$ completion of $k(C)$ wrt the valuation x.

Setting

- Let A be a complete dvr, $K = \mathrm{Frac}(A)$, m_K max. ideal,

$$\begin{tikzcd} K \arrow[r, hook, "\text{fin. Gal}"] & L \\ A \arrow[u, phantom, "\cup"] \arrow[r] & B \arrow[u, phantom, "\cup"] \\ m_K \arrow[u, phantom, "\cup"] \arrow[r] & m_L \arrow[u, phantom, "\cup"] \end{tikzcd}$$

s.t. $k(v_L) = B/m_L \;/\; k(v) = A/m_K$

- Let $p := \mathrm{char}(k(v))$
- $G := \mathrm{Gal}(L/K)$

Def. For $i \geq -1$, $G_i := \{\sigma \in G \mid \sigma \text{ acts trivially on } B/m_L^{i+1}\}$ is the i-th ramif. gp. of G.

$G = G_{-1} \supseteq G_0 \supseteq G_1 \supseteq \cdots$ is the ramification filtration of G in the lower numbering.

Rem. 1) $G_0 = I$ the inertia gp, i.e. the kernel of $G \to \mathrm{Gal}(k(v_L)/k(v)) \to$

2) $G_i = \{\sigma \in G \mid \forall b \in B,\ v_L(\sigma(b) - b) \geq i+1\}$

Fact: $\exists x \in B$ s.t. $B = A[x]$.

$\Downarrow$

$$G_i = \{\sigma \in G \mid v_L(\sigma(x) - x) \geq i+1\} = \{\sigma \in G \mid \sigma(x) - x \in m_L^{i+1}\}$$

Ex: $G_i = 0$ for $i \gg 0$.

Prop. Let $H < G$ be a subgp. Let L^H be its fixed field, so that $\mathrm{Gal}(L/L^H) = H$. Then

$$H_i = G_i \cap H$$

Rem. 1) Now we can assume L/K is totally ramified ($\Leftrightarrow I = G$)

2) Fact: If L/K tot. ramif., $B = A[x]$ with x ~~as an~~ s.t. $v_L(x) = 1$.

Hence, $\cancel{G_i}$ $m_L = (x)$, and

$$G_i = \{\sigma \in G \mid \sigma(x)/x \equiv 1 \pmod{m_L^i}\}$$

(divide by x: $\sigma(x) - x \in (x)^{i+1}$ $\Leftrightarrow$ $\frac{\sigma(x)-x}{x} \in (x)^i$)

Def: (Filtration of $\mathcal{U}_L$).

Let $\mathcal{U}_L := \mathcal{U}_L^0 := B^\times$. For $i > 0$,

$\mathcal{U}_L^i := 1 + m_L^i$ is the gp of i-th units

Rem: $G_i = \{\sigma \in G \mid \sigma(x)/x \in \mathcal{U}_L^i\}$

Prop.

$$G_i/G_{i+1} \hookrightarrow \mathcal{U}_L^i/\mathcal{U}_L^{i+1} \quad \exists \text{ injective hom.}$$
$$\sigma \longmapsto \frac{\sigma(x)}{x}$$

Cor:

1) G_0/G_1 is cyclic of order prime to $p = \mathrm{char}(k)$

2) $p = 0 \Rightarrow G_i = 0$ for $i > 0$

3) $p > 0 \rightsquigarrow$ for $i \geq 1$, G_i are p-gp's, G_i/G_{i+1} abelian p-gp's

4) $G_0 = \underbrace{G_0/G_1}_{\text{cyclic of order prime to } p} \ltimes \underbrace{G_1}_{p\text{-gp.}}$

In particular, G_0 is solvable and G_1 is its unique p-Sylow gp.

Upper numbering:

Notation: for $u \in \mathbb{R}_{\geq -1}$, $G_u := G_{\lceil u \rceil}$

Def.

$$\varphi_{L/K} : [-1, \infty) \longrightarrow [-1, \infty)$$
$$u \longmapsto \int_0^u \frac{dt}{(G_0 : G_t)}$$

~~degree~~ inertia degree

where $(G_0 : G_t) := \begin{cases} \cancel{(G_t : G_0)} \; 1/f & \text{if } t = -1 \\ 0 & \text{if } t \in (-1, 0) \end{cases}$

Rem: If $u \in \mathbb{Z}_{\geq 0}$, $\varphi_{L/K}(u) = \frac{1}{|G_0|} \sum_{i=0}^{u} |G_i| - 1$

Prop. For $H \triangleleft G$, recall ~~we know~~ that $G/H = \mathrm{Gal}(L^H/K)$.
Then

$$G_u H/H = (G/H)_{\varphi_{L/L^H}(u)}$$

Denote $\psi_{L/K} := \varphi_{L/K}^{-1}$

Def: For $v \in \mathbb{R}_{\geq -1}$, $G^v := G_{\psi_{L/K}(v)}$

We get the upper numbering filtration, where the jumps of it are $m \geq -1$ s.t. $G^m \supsetneq G^{m+\varepsilon}$ $\forall \varepsilon > 0$.

Prop. ~~This f.~~ Upper numb. filtr. respects quotients:

$$G^v/(H \cap G^v) = (G/H)^v$$

Sketch:

$$G^v/(H \cap G^v) = G^v H/H := G_{\psi_{L/K}(v)} H/H =$$

$$= (G/H)_{\varphi_{L/L^H}(\psi_{L/K}(v))} =:$$

$$=: (G/H)^{\varphi_{L^H/K}(\varphi_{L/L^H}(\psi_{L/K}(v)))} \overset{?}{=} (G/H)^v$$

Transitivity, associativity

$\square$

Rem. Jumps may not be integral, they are rational.

Thm (Hasse-Arf): If G is abelian, jumps are integer.

Representation theory

Recall: Let G be a finite gp:

A class function from G to an set X, $\varphi: G \to X$, is a function constant on conjugacy classes.

Important example

Let $G := \mathrm{Gal}(L/K)$, with L/K as before (fin. Galois, cdvf).

Let $x \in L$ be a local parameter.

Then
$$i_G : G \longrightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$$
$$\sigma \longmapsto \boxed{v_L(\sigma(x) - x) =: i_G(\sigma)}$$

is a class function:

$$i_G(\tau\sigma\tau^{-1}) = v_L(\tau\sigma\tau^{-1}(x) - x) = v_L(\sigma\tau^{-1}(x) - \tau^{-1}(x)) =$$
$$= i_G(\sigma)$$

~~v_L does~~

$\tau^{-1}(x)$ is again a local parameter

i_G ~~measures~~ gives you the ramification subgps:

$$\boxed{i_G^{-1}([i+1, \infty]) = \{\sigma \mid v_L(\sigma(x) - x) \geq i+1\} = G_i}$$

Ex. Let E be a field, and let $\rho: G \to GL(V)$ be a representation of G on a fin. dim. E-vector sp. V.

The character of ρ, χ_ρ, is

$$\chi_\rho : G \longrightarrow E$$
$$g \longmapsto \chi_\rho(g) := \mathrm{Tr}(\rho(g))$$

Characters are class functions (by linear algebra).

If V is 1-dim., then $\chi_\rho = \rho$.

Rem. Let V_1, V_2 be two repr. of G:

1. $\chi_{V_1 \oplus V_2} = \chi_{V_1} + \chi_{V_2}$

2. $\chi_{V_1 \otimes V_2} = \chi_{V_1} \cdot \chi_{V_2}$

3. $\chi_{V_1^\vee}(g) = \chi_{V_1}(g^{-1})$. Here, if $E = \mathbb{C}$, $\chi_{V_1^\vee}(g) = \overline{\chi_{V_1}(g)}$

Important examples:

- Regular representation.
  - Given G, consider $|G|$-dim. sp. V with basis $\{e_g\}_{g \in G}$,
  then $R_G : G \to V$: $h \cdot e_g := e_{hg}$.
  Let r_G be the character. Then
  $$r_G(g) = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{else} \end{cases}$$
- Trivial representation (of rank 1): $1_G : G \to E$; $g \mapsto 1$.
- Augmentation repr.:
  Starting looking at the quotient
  $$\begin{tikzcd} G \arrow[r, "id"] \arrow[d] & G \arrow[d] \\ V \arrow[r, dashed] & E \end{tikzcd}$$
  $$R_G \longrightarrow 1_G$$
  The kernel is again a representation U_G. if $|G|$ inv. in E, then ~~whose character~~
  ~~$u_G(g) = \{$~~
  $$R_G = U_G \oplus 1_G,$$
  so
  $$u_G(g) = \begin{cases} |G| - 1 & \text{for } g = 1 \\ -1 & \text{else} \end{cases}$$

Let $C_{E,G}$ denote the set of class functions from G to a field E (e.g. characters).

If ~~$|G| \neq$~~ $\mathrm{char}(E) \nmid |G|$, we define
$$\langle \varphi, \psi \rangle_G := \frac{1}{|G|} \sum_{g \in G} \varphi(g) \cdot \psi(g^{-1})$$

and this is a symmetric bilinear form.

Recall if $E = \mathbb{C}$,

Thm: G fin. gp. Then, its irr. characters form a basis of $C_{\mathbb{C},G}$, and this basis is orthonormal wrt $\langle -,- \rangle_G$.

Cor. Over $\mathbb{C}$, a class function φ is a character of G $\Leftrightarrow$ $\varphi = a_1 \chi_1 + \cdots + a_r \chi_r$ with $a_i \in \mathbb{Z}_{\geq 0}$.

Def. Let $\alpha : H \to G$ be a gp homomorphism (e.g. inclusion of a subgp).

$$\begin{tikzcd} C_{\mathbb{C},H} \arrow[r, bend left, "\alpha_*"] & C_{\mathbb{C},G} \arrow[l, bend left, "\alpha^*"] \end{tikzcd}$$

$\alpha^* \varphi := \varphi \circ \alpha$ is the restriction of φ

We define α_* the induced class function as follows:

a) If α inj.,

$$\alpha_* \varphi(g) := \frac{1}{|H|} \sum_{\substack{x \in G \\ xgx^{-1} \in H}} \varphi(xgx^{-1})$$

b) If α surj.,

$$\alpha_* \varphi(g) := \frac{1}{|\ker(\alpha)|} \cdot \sum_{h \mapsto g} \varphi(h)$$

- In general, factor through im. of α.

Rem: 1) this will appear in the pf of the G.O.S., with $\varphi = sw$ the Swan character, and G the Galois gp of a Galois covering of a 1-dim. sm. curve.

2) These functors restrict to characters

Prop (Frobenius reciprocity): For $\alpha : H \to G$, $E = \mathbb{C}$, $C_{\mathbb{C},H} \ni \psi$, $C_{\mathbb{C},G} \ni \varphi$,

$$\langle \psi, \alpha^* \varphi \rangle_H = \langle \alpha_* \psi, \varphi \rangle_G$$

Artin and Swan representations

Let L/K be a fin. Gal. ext. of c.d.v.f. ~~with se~~ that is totally ramified.

Def. The Artin character is the following class function:

$$a_G(g) := \begin{cases} -i_G(g) & \text{if } g \neq 1 \\ \sum_{g' \neq 1} i_G(g') & \text{if } g = 1 \end{cases}$$

Thm. This is a character (of a representation of G over $\mathbb{C}$).

Idea of pf:

For all characters χ of G, we want to show that $\langle a_G, \chi \rangle \in \mathbb{Z}_{\geq 0}$.

- First prove that $\in \mathbb{Q}_{\geq 0}$, because

$$(\heartsuit) \quad \langle a_G, \chi \rangle = \sum_{i \geq 0} \frac{1}{|G : G_i|} (\dim V - \dim V^{G_i}) \qquad (G_i \text{ } i\text{-th ramif. gp})$$

- Left to prove, integrality:
- One reduces to 1-dim. case (Brauer thm + Frob. recipr.), i.e. to proving

$$\langle a_G, \chi' \rangle \in \mathbb{Z} \quad \text{for } \chi' \text{ character of a 1-dim. repr.}$$

- Let $\chi' : G \to \mathbb{C}^\times$ be the repr., and let $G' := \ker(\chi')$.

~~This corresponds to a chain~~

Then $\{1\} < G' < G$ corresponds to a chain

$$L / K' / K$$

Then, $\langle a_G, \chi' \rangle = \quad c' + 1$, where

~~[illegible]~~ is the Herbrand's function and c' is the last break on the ramif. filtration of $\mathrm{Gal}(L/K')$ (with ~~lower~~ upper numbering), i.e. the largest number s.t.

$$\mathrm{Gal}(L/K')^{c'} = (G/H')^{c'} \neq \{1\}.$$

Since G/H' is a subgp of $\mathbb{C}^\times$, is abelian.

Hence Hasse-Arf $\Rightarrow$ c' integral.

Def. $\boxed{\text{Swan character}}$ of G:

$$sw_G := \underbrace{a_G}_{\text{Artin}} - \underbrace{u_G}_{\text{augmentation}}$$

~~where~~

Exercise: sw_G is a character (hint: use ♡)

Fact: ~~if χ is a character of G (fin.) over~~
if χ is the character of a repr. $\rho_{\mathbb{C}}$ of G fin. over $\mathbb{C}$

$$\Downarrow$$

$\rho_{\mathbb{C}}$ factors through a repr. $\rho_{\overline{\mathbb{Q}}}$ over $\overline{\mathbb{Q}}$.

Interview

Q. Can we go further? Is $\rho_{\overline{\mathbb{Q}}}$ realizable over $\mathbb{Q}$?

A. Not in general.

Q. And for $\chi = a_G$ or sw_G.

A. NO, ~~not even~~ a_G may not be realizable even over $\mathbb{R}$.

Q. Can we ~~still~~ do something at all? I want things easier.

A. Ooook, let's try:

in general, $\rho_{\overline{\mathbb{Q}}}$ a repr. over $\overline{\mathbb{Q}}$, and then we have we can base-change to $\overline{\mathbb{Q}}_\ell$

the following:

Thm. Let $\ell \neq \mathrm{char}$(residue field of K), $G = \mathrm{Gal}(L/K)$.

1) Artin and Swan repr. are realizable over $\mathbb{Q}_\ell$.

2) We can go until $\mathbb{Z}_G$ for the Swan repr:

$\exists$ fin. gen. proj. left-$\mathbb{Z}_\ell[G]$-module Sw_G, unique up to isom., s.t. $Sw_G \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ is isomorphic to the Swan repr.

Rem: There is no direct construction of Sw_G known.