Lecture 1

(I) Fundamental group

Let X be a connected ~~noetherian~~ scheme, $\bar{x} \to X$ a geometric point.

Def. (Fibre functor)

étale coverings, i.e. surj. fin. étale

$$\mathrm{Fib}_{\bar{x}} : (\text{ét}/X) \longrightarrow \text{Sets}$$
$$(Y \to X) \longmapsto \mathrm{Hom}_X(\bar{x}, Y)$$

$$\begin{tikzcd} & Y \arrow[d] \\ \bar{x} \arrow[ur, dashed] \arrow[r] & X \end{tikzcd}$$

Rem: $\mathrm{Hom}_X(\bar{x}, Y) \overset{1:1}{\longleftrightarrow} |Y_{\bar{x}}|$, where $Y_{\bar{x}} := Y \times_X \bar{x}$

Def. $\pi_1^{\text{ét}}(X, \bar{x}) := \mathrm{Aut}(\mathrm{Fib}_{\bar{x}})$

Recall: $F: \mathcal{C} \to \mathcal{C}'$ functor.
An autom. of F is a compatible collection of isomorphisms
~~$\{\sigma_{\mathcal{C}}: F(\mathcal{C}) \to F(C), \sigma_C$~~
$\forall C \in \mathcal{C}$, $\sigma_C: F(C) \to F(C)$, σ_C an isom. in $\mathcal{C}'$

Compatibility $\Rightarrow$ $\mathrm{Aut}(\mathrm{Fib}_{\bar{x}}) = \varprojlim_{\substack{Y \to X \\ \text{ét. cov.}}} \mathrm{Fib}_{\bar{x}} Y$

Def. (Galois cov.) Let $Y \to X$ be an ét. cov. It is Galois if
i) Y connected
ii) $\mathrm{Aut}_X(Y) \curvearrowright \mathrm{Fib}_{\bar{x}} Y$ transitive.

$$\begin{tikzcd} & Y \arrow[r] \arrow[d] & Y \arrow[dl] \\ \bar{x} \arrow[ur, dashed] \arrow[r] & X & \end{tikzcd}$$

In this case, $G_Y := \mathrm{Aut}_X(Y)$ is the Galois gp of $Y \to X$, ~~G_Y~~

Prop 1. There are Galois closures:

$$\begin{tikzcd} P' \arrow[r, dashed, "\exists"] \arrow[dr] \arrow[ddr, "\mathrm{Gal.}"'] & P \arrow[d, dashed, "\exists\,\Pi"] \\ & Y \arrow[d, "\text{ét. cov.}"] \\ & X \end{tikzcd}$$

s.t. $\begin{matrix} P \\ \downarrow \mathrm{Gal.} \\ X \end{matrix}$ and

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Galois coverings determine fund. gp:

For each $P_\alpha \to X$ Gal. covering of X, choose a lift of $\bar{x} \to X$, p_α

$$\begin{tikzcd} & P_\alpha \arrow[d] \\ \bar{x} \arrow[ur, dashed, "p_\alpha"] \arrow[r] & X \end{tikzcd}$$

$\{P_\alpha, p_\alpha\}$ is a directed system, and moreover, fixing p_α's imposes that there is at most one $(P_\beta, p_\beta) \xrightarrow{\phi_{\beta,\alpha}} (P_\alpha, p_\alpha)$

Hence, $\left((P_\alpha, p_\alpha), \phi_{\beta,\alpha}\right)$ is a proj. system, and

Prop 2. $\pi_1^{\text{ét}}(X, \bar{x}) \simeq \varprojlim \mathrm{Aut}_X(P_\alpha)^{op}$

Ex. If $X = \mathrm{Spec}(K)$,

- Ét. cov. are fin. sep. extensions of K
- Gal cov. fin. Galois
- ~~$\pi_1^{\text{ét}}(X, \bar{x}$~~ $\bar{x} \to K$ corresponds to an emb. $K \hookrightarrow \Omega$ alg. closed
- ~~Inf~~ Prop 1. is just that given a sep. ~~closure,~~ ext., there exists a Gal. closure.
- Prop 2: $\varprojlim_{\substack{L/K \\ \text{fin. sep.}}} L = \varprojlim_{\substack{L/K \\ \text{fin. Gal.}}} L$.

Thm $\quad \mathrm{Fib}_{\bar{x}} : (\text{ét}/X) \longrightarrow \underline{\text{Sets}}$

Ex. $\quad Y \to X$ irr. var. over k $\quad$ Galois cov. $\longleftrightarrow$ $K(Y)/K(X)$ fin. Galois AND if is unramified for all valuations in $\mathcal{O}_X(X)$.

Hence, X irr. $\quad$ Prop. 2 $\rightsquigarrow$ $\pi_1^{\text{ét}}(X, \bar{\eta}) \simeq \mathrm{Gal}(K(X)^{unr}/K(X))$,

$K(X)^{unr} :=$ maximal $\quad \ldots \quad$ sep. ext. unr at $\mathcal{O}_X(X)$.

Thm. Let X be a conn. scheme, $\bar{x} \longrightarrow X$ a geom. point.

$$\mathrm{Fib}_{\bar{x}} : (\text{ét}/X) \longrightarrow \underline{F\text{ Sets} + \text{action of } \pi_1^{\text{ét}}(X, \bar{x})}$$
$$(Y \to X) \longmapsto Y_{\bar{x}}$$

continuous

is an equiv. of categories (!).

ℓ-adic sheaves

Recall

Let X be separated and noetherian.

Recall

Let $\mathcal{F}$ be an ét sheaf on X (i.e. a contravariant functor $\mathcal{F} : (\text{ét}/X) \to \{\underline{A\text{-mod}}\}$ satisfying the sheaf axioms in $X_{\text{ét}}$)

Then

$$\mathcal{F}_{\bar{x}} := \varinjlim_{\substack{\bar{x} \to \mathcal{U} \\ \mathcal{U} \text{ ét. cov.}}} \mathcal{F}(\mathcal{U})$$

Given $\eta : X \to Y$, then $\eta_* \mathcal{F} : \begin{pmatrix} V \\ \downarrow \\ Y \end{pmatrix} \mapsto \mathcal{F}(V \times_Y X)$

Category theory $\Rightarrow$ $\exists$ left adjoint: $\pi^* : (\acute{e}t/Y) \to (\acute{e}t/X)$ s.t.

$$\mathrm{Hom}_{X_{\acute{e}t}}(\pi^* \mathcal{G}, \mathcal{F}) \xrightarrow{1.1.} \mathrm{Hom}_{Y_{\acute{e}t}}(\mathcal{G}, \pi_* \mathcal{F})$$

Locally, $(\pi^* \mathcal{G})_{\bar{x}} = \mathcal{G}_{(\pi \circ \bar{x})}$ $\qquad \bar{x} \to X \xrightarrow{\pi} Y$

We denote also $\mathcal{G}|_X := \pi^* \mathcal{G}$

Def. $\mathcal{F}$ loc. constant if ~~exists~~ $\exists$ $\{U_i \to X\}$ s.t. $\mathcal{F}|_{U_i} = M_{U_i}$ with stalk M

For cohom.

Extension by 0: $\quad \overset{\mathcal{F}}{j : U \to X}, \quad i : Z \to X$, $Z = X \setminus U$

$$j_! \mathcal{F} := \ker(i^* j_* \mathcal{F} \to i_* i^* j_* \mathcal{F}) \qquad i^* j_* \mathcal{F} \to i_* i^* j_* \mathcal{F}$$

Locally, $(j_! \mathcal{F})_{\bar{x}} = \begin{cases} \mathcal{F}_{\bar{x}} & \text{if } \bar{x} \in U \\ 0 & \text{else.} \end{cases}$

Def. Let A be noeth ring that is torsion (i.e. $mA = 0$ for some $m \in \mathbb{N}_{>0}$).

Let $\mathcal{F}$ be a sheaf of A-mod. on $X_{\acute{e}t}$. Then $\mathcal{F}$ is constructible if there exist finite type A-modules $M_1, \dots, M_n$ and locally closed $X_1, \dots, X_n \subset X$ s.t.

(i) $X = \bigsqcup X_i$

(ii) $\mathcal{F}|_{X_i}$ are locally constant with stalks M_i.

Rem. Loc. constant $\Rightarrow$ constructible
$\not\Leftarrow$

X scheme

Def. Let R be a local complete dvr with maximal ideal m and residue field of char. $\ell > 0$ (think on $\mathbb{Z}_\ell = \varprojlim \mathbb{Z}/\ell^n$)

1) A constructible R-sheaf on X is a proj. system of R-modules $\mathcal{F} := (\mathcal{F}_n)_{n \geq 1}$, each $\mathcal{F}_n$ on $X_{\acute{e}t}$, s.t.

a) $\mathcal{F}_n$ is a constructible R/m^n-mod. s.t. $m^n \mathcal{F}_n = 0$

b) For all $n \geq 1$, $\mathcal{F}_n = \mathcal{F}_{n+1} \otimes_{R/m^{n+1}} R/m^n$

2) lisse R-sheaf is a const. R-sheaf $\mathcal{F} = (\mathcal{F}_n)$ s.t. each $\mathcal{F}_n$ is a locally constant R/m^n-

Def. Let X scheme with ℓ invertible, and fix $\overline{\mathbb{Q}}_\ell / \mathbb{Q}_\ell$.

- constructible E-sheaf, $E/\mathbb{Q}_\ell$ finite, is a constructible $\mathcal{O}_E$-sheaf $\mathcal{F}$ tensored with E.

  $\mathcal{F} \otimes E$ is lisse if $\exists \{U_i \to X\}$ and lisse $\mathcal{O}_E$-sheaves $\mathcal{F}_i$ s.t. $\mathcal{F}|_{U_i} \otimes E \simeq \mathcal{F}_i \otimes E$.

- constructible $\overline{\mathbb{Q}}_\ell$-sheaf: is an object on the limit of categories of constructible E-sheaves, $\forall\, E/\mathbb{Q}_\ell$ fin.

  If E'/E,

$$\mathcal{F} \otimes_{\mathcal{O}_E} E \longmapsto (\mathcal{F} \otimes_{\mathcal{O}_E} E) \otimes_E E' = \mathcal{F} \otimes_{\mathcal{O}_{E'}} E'$$

Convention: the ring A is an ℓ-adic coeff ring if: $\begin{cases} \mathcal{O}_E / m^n, & n \geq 1 \\ \mathcal{O}_E \\ E \\ \overline{\mathbb{Q}}_\ell \end{cases}$

of $\mathcal{F}$

Let k be a perf. field of char. p, $\ell \neq p$ ~~and~~ a prime and A an ~~an~~ ℓ-adic ~~sheaf~~ coeff. ring.

Let $\mathcal{F}$ be a constructible A-sheaf, so that

$$\mathcal{F} = \underbrace{(\mathcal{F}_n)}_{\text{constr. } \mathcal{O}_E\text{-sheaf}} \otimes_{\mathcal{O}_E} A$$

<u>Def</u>: $\mathcal{F}_{\bar{x}} := \left\{ \left(\varprojlim \mathcal{F}_{n,\bar{x}} \right) \otimes_{\mathcal{O}_E} A \right.$.

<u>Rem</u>: $\mathcal{F}$ lisse $\Rightarrow$ $\mathcal{F}_{\bar{x}}$ is a finite type A-module.

<u>Def</u>. $\mathcal{F}$ is <u>free</u> if $\mathcal{F}_{\bar{x}}$ are free A-modules.

<u>Def</u>. Let M be a fin. gen. A-module, X a sep noeth. conn. scheme, $\bar{x} \to X$ a geom. point.

- If $A \neq \overline{\mathbb{Q}}_\ell$, an <u>$A$-repr. of $\pi_1^{ét}(X, \bar{x})$</u> is a continuous gp hom.
$$\pi_1^{ét}(X, \bar{x}) \to \mathrm{Aut}_A(M)$$
- If $A = \overline{\mathbb{Q}}_\ell$, a $\overline{\mathbb{Q}}_\ell$-repr. of $\pi_1^{ét}(X, \bar{x})$ is a cts gp hom.
$$\pi_1^{ét}(X, \bar{x}) \to \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(M)$$ <u>coming</u> from an E-repr, with $E/\mathbb{Q}_\ell$ fin.

<u>Thm</u>. X conn., $\bar{x} \to X$ geom. pt, A ℓ-adic coeff. ring.

$$\left(\text{lisse } A\text{-sheaves on } X\right) \longrightarrow \left(A\text{-repr. of } \pi_1^{ét}(X, \bar{x})\right)$$
$$\mathcal{F} \longmapsto \mathcal{F}_{\bar{x}}$$

The action is as follows:

For $\sigma \in \pi_1^{ét}(X, \bar{x})$ and $\begin{matrix} P_\alpha \\ \downarrow \\ X \end{matrix}$, we have $\sigma^*: F(P_\alpha) \to F(P_\alpha)$

If $U \to X$ ét. neighb., $U_\alpha := U \times P_\alpha$, we have $\mathcal{F}(U) \to \mathcal{F}(U_\alpha) \to \mathcal{F}(U_\alpha) \to \mathcal{F}_{\bar{x}}$

This system is compatible as we change $\mathcal{U}$, so we obtain

$\sigma^*: \mathcal{F}_{\bar{x}} \longrightarrow \mathcal{F}_{\bar{x}}$

and this is the representation.

Furthermore, if $X' \to X$ is a finite Galois cover with Galois gp G, we have an equivalence

$$(\text{lisse } A\text{-sheaves on } X, \text{ constant on } X') \overset{1:1}{\longleftrightarrow} (\text{fin. gen. } A[G]\text{-modules})$$

Action of a gp on a scheme:

We say that $G \curvearrowright X$ if there is a hom. $G \to \mathrm{Aut}(X)$.

- $G \curvearrowright X$ admissibly if X is a union of open affines $\mathcal{U}$ s.t. G acts on $\mathcal{U}$. ~~In this case,~~

  E.g.: G finite $\curvearrowright X$ quasi-proj. over a field $\Rightarrow G \curvearrowright X$ adm.

- If $G \curvearrowright X$ is admissible, we can form the quotient $\pi: X \to X/G$ by glueing $\mathcal{U}/G := \mathrm{Spec}(A^G)$. Then,

$$\mathrm{Hom}(X, Y)^G = \mathrm{Hom}(X/G, Y) \quad \forall Y.$$

Def. Let $G \curvearrowright X$ admissibly, $\mathcal{F}$ a sheaf of A-modules on $X_{\acute{e}t}$ together with morph.

$$\mathcal{F}(\sigma): \mathcal{F} \to \sigma^* \mathcal{F}$$

$\mathcal{F} \to \tau^* \mathcal{F}$

$\tau^* \mathcal{F} \to \tau^* \sigma^* \mathcal{F}$

s.t. $\mathcal{F}(1_G) = id_{\mathcal{F}}$, $\mathcal{F}(\sigma\tau) = \tau^*(\mathcal{F}(\sigma)) \circ \mathcal{F}(\tau)$

Then $\mathcal{F}$ is a sheaf with a G-action.

Rem. Given $\pi: X \to X/G$, then G has trivial action on X/G and acts on $\pi_* \mathcal{F}$. Hence:

Def.
$$(\pi_* \mathcal{F})^G: \underset{X/G}{\mathcal{U}} \mapsto \{ a \in \mathcal{F}(\mathcal{U} \times_{X/G} X) \mid \mathcal{F}(\sigma)(a) = a \quad \forall \sigma \in G \}$$

Similarly, if ℓ is inv. in X, ~~and~~ A an ℓ-adic coeff. ring and $\mathcal{F} = (\mathcal{F}_n) \otimes_{\mathcal{O}_E} A$ a constructible A-sheaf;

$\mathcal{F}^G := (\mathcal{F}_n{}^G) \otimes_{\mathcal{O}_E} A$

Even more in our thm: $\mathcal{F} \longmapsto \mathcal{F}_{\bar{x}}$

- If A is finite, (fin. gen. $A[G]$-mod)

$$(\pi_* M_{X'})^G \longleftarrow\!\shortmid M$$

- If A infinite, $M = N \otimes_{\mathcal{O}_E} A$ and

$$\left((\pi_* N_{X'})^G \otimes_{\mathcal{O}_E} \mathcal{O}_E/m^n\right)_n \otimes_{\mathcal{O}_E} A \longleftarrow\!\shortmid M$$

Wild ramification of an ℓ-adic sheaf.

Setting:

- C smooth proper geom. conn. curve / k perfect, char $= p > 0$.
  $\cup$
  $\mathcal{U}$ affine open subset.
- $\ell \neq p$ prime number
- $K := k(\mathcal{U})$, fix $\bar{K}/K$. ~~η~~ $\eta \to C$ gen. point, $\bar{\eta} \to C$ geometric point $\Rightarrow$. $\bar{K}/K^{sep}/K$
- If $x \in C$ closed, $K_x :=$ completion of K wrt x. $\rightsquigarrow$ complete d.v. field

  Choose $i_x : K^{sep} \to K_x^{sep}$ over K.
- $G := Gal(K^{sep}/K) \supset D_x \supset I_x \supset P_x$

  $\cup$ inertia, wild inertia, is pro-p-gp

  $Gal(K_x^{sep}/K_x) \supset I_x$ (ine.)

Let A be an l-adic coeff. ring, $\mathcal{F}$ a free lisse A-module on $\mathcal{U}$.

Then $\Rightarrow$ $\mathcal{F}$ corresponds to $\pi_1^{ét}(\mathcal{U},\bar{\eta}) \to \mathcal{F}_{\bar{\eta}}$.

Recall: $\mathrm{Gal}(k^{unr}(\mathcal{U})/k(\mathcal{U})) \simeq \pi_1^{ét}(\mathcal{U},\bar{\eta})$

We have a surjection $G \to \mathrm{Gal}(k^{unr}(\mathcal{U})/k(\mathcal{U}))$

~~and~~ Let $x \in C$ be a closed point, then we have

$$P_x \hookrightarrow D_x \hookrightarrow G \to \mathrm{Gal}(k^{unr}(\mathcal{U})/k(\mathcal{U})) \xrightarrow{\sim} \pi_1^{ét}(\mathcal{U},\bar{\eta}) \to \mathcal{F}_{\bar{\eta}}$$

Fact: if $P_x \to \mathcal{F}_{\bar{\eta}}$ is the repr. of a pro-p-gp, it factors through a finite quotient of P_x.

Fact: If $P_x \to \mathcal{F}_{\bar{\eta}}$ factors through a fin. quotient, there exists a break decomposition

$$\mathcal{F}_{\bar{\eta}} = \bigoplus_{t_x \in \mathbb{R}_{\geq 0}} \mathcal{F}_{\bar{\eta}}(t_x) \qquad \text{(the } x \text{ is to remember the choice)}$$

with nice properties.

Def (Swan conductor): $\mathrm{Swan}_x \mathcal{F}_{\bar{\eta}} := \sum_{t_x \geq 0} t_x \cdot \mathrm{rank}(\mathcal{F}_{\bar{\eta}}(t_x)) \in \mathbb{R}$

Def. The wild ramification of $\mathcal{F}$ at x is

$$\mathrm{Swan}_x(\mathcal{F}) := \mathrm{Swan}_x(\mathcal{F}_{\bar{\eta}}).$$

$\boxed{\text{G.O.S.}}$ Let $\mathcal{F}$ be a lisse $\overline{\mathbb{Q}}_\ell$-sheaf on $\mathcal{U}$. Then

$$\chi_c(\overline{\mathcal{U}},\mathcal{F}) = \mathrm{rk}(\mathcal{F}) \cdot \chi_c(\overline{\mathcal{U}},\overline{\mathbb{Q}}_\ell) - \sum_{x \in C \setminus \mathcal{U}} [k(x):k] \cdot \mathrm{Swan}_x(\mathcal{F})$$

$\overline{\mathcal{U}} := \mathcal{U} \otimes \bar{k}$, $\overline{\mathbb{Q}}_\ell = (\mathbb{Z}/\ell^n\mathbb{Z}) \otimes \ldots \overline{\mathbb{Q}}_\ell$