

Lecture 1

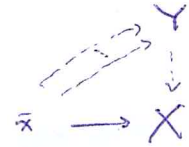
I Fundamental group

Let X be a connected ~~noetherian~~ scheme, $\bar{x} \rightarrow X$ a geometric point.

Def. (Fibre functor) \nearrow étale coverings, i.e. surj. fin. étale

$$\text{Fib}_{\bar{x}} : (\text{ét}/X) \longrightarrow \text{Sets}$$

$$(Y \rightarrow X) \longmapsto \text{Hom}_X(\bar{x}, Y)$$



Rem: $\text{Hom}_X(\bar{x}, Y) \xrightarrow{1:1} |Y_{\bar{x}}|$ where $Y_{\bar{x}} := Y \times_X \bar{x}$

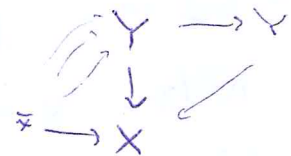
Def. $\pi_1^{\text{ét}}(X, \bar{x}) := \text{Aut}(\text{Fib}_{\bar{x}})$

Recall: $F: \mathcal{C} \rightarrow \mathcal{C}'$ functor.
 An automorphism of F is a compatible collection of isomorphisms
 ~~$\{\sigma_C: F(C) \rightarrow F(C), \sigma_C\}$~~
 $\forall C \in \mathcal{C}, \sigma_C: F(C) \rightarrow F(C), \sigma_C$ an isom. in \mathcal{C}'

Compatibility \Rightarrow $\text{Aut}(\text{Fib}_{\bar{x}}) = \varprojlim_{\substack{Y \rightarrow X \\ \text{ét. cov.}}} \text{Fib}_{\bar{x}} Y$

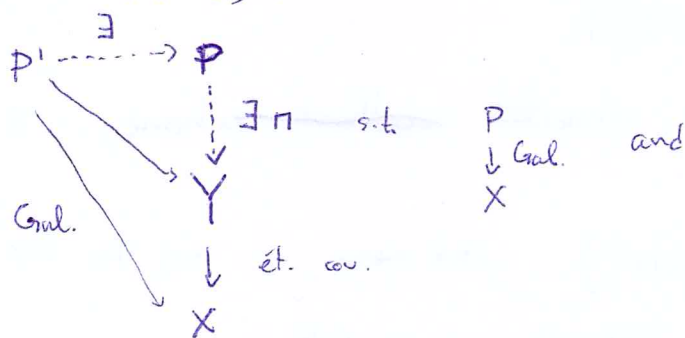
Def. (Galois cov.) \curvearrowright Let $Y \rightarrow X$ be an ét. cov. It is Galois if

- (i) Y connected
- (ii) $\text{Aut}_X(Y) \curvearrowright \text{Fib}_{\bar{x}} Y$ transitive.



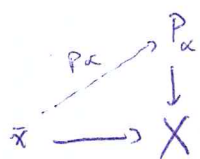
In this case, $G_Y := \text{Aut}_X(Y)$ is the Galois group of $Y \rightarrow X$, $G_Y =$

Prop 1. There are Galois closures:



Galois coverings determine fund. gp:

For each $P_\alpha \rightarrow X$ Gal. covering of X , choose a lift of $\bar{x} \rightarrow X$, p_α



$\{P_\alpha, p_\alpha\}$ is a directed system, and moreover, fixing p_α 's imposes that there is at most one

$$(P_\beta, p_\beta) \xrightarrow{\phi_{\beta,\alpha}} (P_\alpha, p_\alpha)$$

Hence, $(P_\alpha, p_\alpha), \phi_{\beta,\alpha}$ is a proj. system, and

Prop 2. $\pi_1^{ét}(X, \bar{x}) \cong \varprojlim \text{Aut}_X(P_\alpha)^{p_\alpha}$

Ex. If $X = \text{Spec}(K)$,

- Ét. cov. are fin. sep. extensions of K

- Gal. cov. fin. Galois

- $\pi_1^{ét}(X, \bar{x}) \cong \bar{x} \rightarrow K$ corresponds to an emb. $K \hookrightarrow \Omega$ alg. closed

- Prop 1 is just that given a sep. ~~closure~~ ext., there exists a Gal. closure.

- Prop 2: $\varprojlim_{L/K} L = \varprojlim_{L/K} L$
fin. sep. fin. Gal.

Thm $\text{Fib}_{\bar{x}} : (\text{ét}/X) \rightarrow \underline{\text{Sets}}$

Ex. $Y \rightarrow X$ Galois cov. $\leftrightarrow K(Y)/K(X)$ fin. Galois AND
 irr. var. over k it is unramified for all valuations
 in $\mathcal{O}_X(X)$.

Hence, X irr. Prop. 2 $\rightarrow \pi_1^{\text{ét}}(X, \bar{x}) \cong \text{Gal}(K(X)^{\text{unr}}/K(X)$,
 $K(X)^{\text{unr}}$:= maximal sep. ext. unr at $\mathcal{O}_X(X)$.

Thm. Let X be a conn. scheme, $\bar{x} \rightarrow X$ a geom. point.

$\text{Fib}_{\bar{x}} : (\text{ét}/X) \rightarrow \underline{\text{F Sets + action of } \pi_1^{\text{ét}}(X, \bar{x})}$
 $(Y \rightarrow X) \mapsto Y_{\bar{x}}$ continuous

is an equiv. of categories (!).

l-adic sheaves

Recall

Let X be separated and noetherian.

Recall

Let \mathcal{F} be an ét sheaf on X (i.e. a contravariant functor

$\mathcal{F} : (\text{ét}/X) \rightarrow \underline{\mathbb{Z} A\text{-mod}}$ satisfying the sheaf axioms in $X_{\text{ét}}$

Then

$\mathcal{F}_{\bar{x}} := \varinjlim_{\substack{U \rightarrow X \\ U \text{ ét. cov.}}} \mathcal{F}(U)$

Given $\eta : X \rightarrow Y$, then $\eta_* \mathcal{F} : \left(\begin{smallmatrix} V \\ \downarrow \\ Y \end{smallmatrix} \right) \mapsto \mathcal{F}(V \times_Y X)$

Category theory $\Rightarrow \exists$ left adjoint: $\pi^*: (\text{ét}/Y) \rightarrow (\text{ét}/X)$ s.t.

$$\text{Hom}_{X_{\text{ét}}}(\pi^* G, F) \xrightarrow{\text{i.l.}} \text{Hom}_{Y_{\text{ét}}}(G, \pi_* F)$$

Locally, $(\pi^* G)_{\bar{x}} = G_{(\pi \circ \bar{x})} \quad \bar{x} \rightarrow X \xrightarrow{\pi} Y$

We denote also $G|_X := \pi^* G$

Def. F loc. constant if ~~there~~ $\exists \{U_i \rightarrow X\}$ s.t. $F|_{U_i} = M_{U_i}$
with stalk M

For cohom.

Extension by 0: $j: U \rightarrow X, \quad i: Z \rightarrow X$

$$j! F := \ker(i^* j_* F \rightarrow i_* i^* j_* F) \quad i^* j_* F \rightarrow i_* i^* j_* F$$

Locally, $(j! F)_{\bar{x}} = \begin{cases} F_{\bar{x}} & \text{if } \bar{x} \in U \\ 0 & \text{else.} \end{cases}$

Def. Let A be noeth ring that is torsion (i.e. $m A = 0$ for some $m \in \mathbb{N}_{>0}$).

Let F be a sheaf of A -mod. on $X_{\text{ét}}$. Then F is constructible if there exist finite type A -modules

M_1, \dots, M_n and locally closed $X_1, \dots, X_n \subset X$ s.t.

(i) $X = \bigsqcup X_i$

(ii) $F|_{X_i}$ are locally constant with stalks M_i .

Rem. Loc. constant \Rightarrow constructible
 \Leftarrow

X scheme
Def. Let R be a ^{local} complete DVR with maximal ideal m and residue field of char. $l \neq 0$ (think on $\mathbb{Z}_l = \varprojlim \mathbb{Z}/l^n$)

1) constructible R -sheaf on X is a proj. system of R -modules $\mathcal{F} := (\mathcal{F}_n)_{n \geq 1}$, each \mathcal{F}_n on $X_{\text{ét}}$, s.t.

a) \mathcal{F}_n is a constructible R/m^n -mod. s.t. $m^n \mathcal{F}_n = 0$.

b) For all $n \geq 1$, $\mathcal{F}_n = \mathcal{F}_{n+1} \otimes_{R/m^{n+1}} R/m^n$

2) lisse R -sheaf is a const. R -sheaf $\mathcal{F} = (\mathcal{F}_n)$ s.t.

each \mathcal{F}_n is a locally constant R/m^n -

Def. Let X scheme with l invertible, and fix $\overline{\mathbb{Q}_l}/\mathbb{Q}_l$.

• constructible E -sheaf, E/\mathbb{Q}_l finite, is a constructible \mathbb{Q}_l -sheaf \mathcal{F} tensored with E .

$\mathcal{F} \otimes E$ is lisse if $\exists \{U_i \rightarrow X\}$ and lisse \mathbb{Q}_l -sheaves \mathcal{F}_i s.t.
 $\mathcal{F}|_{U_i} \otimes E \cong \mathcal{F}_i \otimes E$.

• constructible $\overline{\mathbb{Q}_l}$ -sheaf: is an object on the limit of categories of constructible E -sheaves, $\forall E/\mathbb{Q}_l$ fin.

if E'/E ,

$$\mathcal{F} \otimes_{\mathbb{Q}_l} E \mapsto (\mathcal{F} \otimes_{\mathbb{Q}_l} E) \otimes_{\mathbb{Q}_l} E' = \mathcal{F} \otimes_{\mathbb{Q}_l} E'$$

Convention: the ring A is an l -adic coeff. ring if: $\begin{cases} \mathbb{Q}_l/m^n, n \geq 1 \\ \mathbb{Q}_l \\ E \\ \overline{\mathbb{Q}_l} \end{cases}$

Let k be a perf. field of char. p , $l \neq p$ and a prime
and A an l -adic ~~sheaf~~ coeff. ring.

Let \mathcal{F} be a constructible A -sheaf, so that

$$\mathcal{F} = \underbrace{(\mathcal{F}_n)}_{\text{comp. } \mathcal{O}_E\text{-sheaf}} \otimes_{\mathcal{O}_E} A$$

Def. $\mathcal{F}_{\bar{x}} := \left\{ \left(\varprojlim \mathcal{F}_n, \bar{x} \right) \otimes_{\mathcal{O}_E} A \right.$

Rem: \mathcal{F} lisse $\Rightarrow \mathcal{F}_{\bar{x}}$ is a finite type A -module.

Def. \mathcal{F} is free if $\mathcal{F}_{\bar{x}}$ are free A -modules.

Def. Let M be a fin. gen. A -module, X a sep
noeth. conn. scheme, $\bar{x} \rightarrow X$ a geom. point.

• If $A \neq \bar{\mathcal{O}}_E$, an A -repr. of $\pi_1^{\text{ét}}(X, \bar{x})$ is a continuous gp
hom.

$$\pi_1^{\text{ét}}(X, \bar{x}) \rightarrow \text{Aut}_A(M)$$

• If $A = \bar{\mathcal{O}}_E$, a $\bar{\mathcal{O}}_E$ -repr. of $\pi_1^{\text{ét}}(X, \bar{x})$ is a cts gp
hom.

$\pi_1^{\text{ét}}(X, \bar{x}) \rightarrow \text{Aut}_{\bar{\mathcal{O}}_E}(M)$ coming from an E -repr,
with $E/\bar{\mathcal{O}}_E$ fin.

Thm. X conn., $\bar{x} \rightarrow X$ geom. pt, A l -adic coeff. ring.

$$(\text{lisse } A\text{-sheaves on } X) \longrightarrow (A\text{-repr. of } \pi_1^{\text{ét}}(X, \bar{x}))$$

$$\mathcal{F} \longmapsto \mathcal{F}_{\bar{x}}$$

The action is as follows:

For $\sigma \in \pi_1^{\text{ét}}(X, \bar{x})$ and P_α , we have $\sigma^*: F(P_\alpha) \rightarrow F(P_\alpha)$

If $U \rightarrow X$ ^{neighb.} _{ét.}, $U_\alpha := U \times P_\alpha$, we have $F(U) \rightarrow F(U_\alpha) \rightarrow F(U_\alpha) \rightarrow \mathcal{F}_{\bar{x}}$

(6)

This system is compatible as we change U , so we obtain

$$\sigma^*: \mathcal{F}_{\bar{x}} \rightarrow \mathcal{F}_{\bar{x}}$$

and this is the representation.

Furthermore, if $X' \rightarrow X$ is a finite Galois cover with Galois gp G , we have an equivalence

$$(\text{fibre } A\text{-sheaves on } X, \text{ constant on } X') \xrightarrow{\sim} (\text{fin. gen. } A[G]\text{-modules})$$

Action of a gp on a scheme:

We say that $G \curvearrowright X$ if there is a hom. $G \rightarrow \text{Aut}(X)$.

- $G \curvearrowright X$ admissibly if X is a union of open affines U s.t. G acts on U .

~~in this case,~~

E.g.: G finite $\curvearrowright X$ quasi-proj. over a field $\Rightarrow G \curvearrowright X$ adm.

- If $G \curvearrowright X$ is admissible, we can form the quotient $\pi: X \rightarrow X/G$ by gluing $U/G \cong \text{Spec}(A^G)$. Then,

$$\text{Hom}(X, Y)^G = \text{Hom}(X/G, Y) \quad \forall Y.$$

Def. Let $G \curvearrowright X$ admissibly, \mathcal{F} a sheaf of A -modules on $X_{\text{ét}}$ together with morph.

$$\mathcal{F}(\sigma): \mathcal{F} \rightarrow \sigma^* \mathcal{F}$$

$$\mathcal{F} \rightarrow \tau^* \mathcal{F}$$

$$\tau^* \mathcal{F} \rightarrow \tau^* \sigma^* \mathcal{F}$$

s.t. $\mathcal{F}(1_G) = \text{id}_{\mathcal{F}}$, $\mathcal{F}(\sigma\tau) = \tau^*(\mathcal{F}(\sigma)) \circ \mathcal{F}(\tau)$

Then \mathcal{F} is a sheaf with a G -action.

Rem. Given $\pi: X \rightarrow X/G$, then G has trivial action on X/G and acts on $\pi_* \mathcal{F}$. Hence:

Def.

$$(\pi_* \mathcal{F})^G: \mathcal{U} \xrightarrow{\pi/G} \left\{ a \in \mathcal{F}(\mathcal{U} \times_{X/G} X) \mid \mathcal{F}(\sigma)(a) = a \quad \forall \sigma \in G \right\}$$

Similarly, if k is inv. in X , and A an l -adic coeff. ring
 and $\mathcal{F} = (\mathcal{F}_n) \otimes_{\mathbb{O}_E} A$ a constructible A -sheaf;
 $\tilde{\mathcal{F}}^G := (\mathcal{F}_n^G) \otimes_{\mathbb{O}_E} A$

Even more in our thm: $\mathcal{F} \longleftrightarrow \mathcal{F}_{\bar{x}}$

If A is finite, $(n_* M_{X^i})^G \longleftrightarrow M$ (fin. gen. $A[G]$ -mod)

If A infinite, $M = N \otimes_{\mathbb{O}_E} A$ and

$$\left((n_* N_{X^i})^G \otimes_{\mathbb{O}_E} \mathbb{O}_E / \mathfrak{m}^n \right) \otimes_{\mathbb{O}_E} A \longleftrightarrow M$$

Wild ramification of an l -adic sheaf.

Setting:

C smooth proper geom. conn. curve / k perfect, char = $p > 0$

U affine open subset.

$l \neq p$ prime number

$K := k(U)$, fix \bar{K}/K . $\eta \rightarrow C$ gen. point,

$\bar{\eta} \rightarrow C$ geometric point \uparrow . $\bar{K}/K^{sep}/K$

If $x \in C$ closed, $K_x :=$ completion of K wrt x . \rightarrow complete 'dv. field

Choose $\iota_x: K^{sep} \rightarrow K_x^{sep}$ over K .

$G := \text{Gal}(K^{sep}/K) \supset D_x \supset I_x \supset P_x$

$\text{Gal}(K_x^{sep}/K_x) \supset I_x^{in}$ inertia, wild inertia, is pro- p -gp

Let A be an ℓ -adic coeff. ring, \mathcal{F} a free lisse A -module on U .

Then $\Rightarrow \mathcal{F}$ corresponds to $\pi_1^{\text{ét}}(U, \bar{\eta}) \rightarrow \mathcal{F}_{\bar{\eta}}$.

Recall: $\text{Gal}(k^{\text{unr}}(U)/k(U)) \simeq \pi_1^{\text{ét}}(U, \bar{\eta})$

We have a surjection $G \rightarrow \text{Gal}(k^{\text{unr}}(U)/k(U))$

~~Let~~ Let $x \in C$ be a closed point, then we have

$$P_x \hookrightarrow D_x \hookrightarrow G \rightarrow \text{Gal}(k^{\text{unr}}(U)/k(U)) \xrightarrow{\simeq} \pi_1^{\text{ét}}(U, \bar{\eta}) \rightarrow \mathcal{F}_{\bar{\eta}}$$

Fact: If $P_x \rightarrow \mathcal{F}_{\bar{\eta}}$ is the repr. of a pro- ℓ -gp, it factors through a finite quotient of P_x .

Fact: If $P_x \rightarrow \mathcal{F}_{\bar{\eta}}$ factors through a fin. quotient, there exists a break decomposition

$$\mathcal{F}_{\bar{\eta}} = \bigoplus_{t_x \in \mathbb{R}_{\geq 0}} \mathcal{F}_{\bar{\eta}}(t_x) \quad (\text{the } x \text{ is to remember the choice})$$

with nice properties.

Def (Swan conductor): $\text{Swan}_x \mathcal{F}_{\bar{\eta}} := \sum_{t_x \geq 0} t_x \cdot \text{rank}(\mathcal{F}_{\bar{\eta}}(t_x)) \in \mathbb{R}$

Def: The wild ramification of \mathcal{F} at x is

$$\text{Swan}_x(\mathcal{F}) := \text{Swan}_x(\mathcal{F}_{\bar{\eta}}).$$

G.O.S. Let \mathcal{F} be a lisse $\overline{\mathbb{Q}_\ell}$ -sheaf on U . Then

$$\chi_c(\bar{U}, \mathcal{F}) = \text{rk}(\mathcal{F}) \cdot \chi_c(\bar{U}, \overline{\mathbb{Q}_\ell}) - \sum_{x \in C \cup U} [k(x):k] \cdot \text{Swan}_x(\mathcal{F}) \quad (9)$$

$\bar{U} := U \otimes_k \bar{k}$, $\overline{\mathbb{Q}_\ell} = (\mathbb{Z}/\ell^n \mathbb{Z}) \propto \bar{\mathbb{Q}_\ell}$

