MOTIVES FOR PERIODS

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*Notes by Pedro A. Castillejo. They are not double checked, be aware of mistakes!

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Minicourses

JOSEPH AYOUB – *Triangulated categories of motives and the Kontsevich-Zagier conjecture*

I will recall the construction of the triangulated categories of motives and discuss various related topics (the Betti and de Rham realisations, the rigid analytic variant, nearby motives, etc.). Then, I will recall the construction of the motivic Galois group and the torsor of motivic periods, and formulate the Kontsevich-Zagier conjecture on periods in this setting. Finally, I will formulate a geometric version of the Kontsevich-Zagier conjecture and explain its proof.

CLÉMENT DUPONT – *Mixed Tate motives and multiple zeta values*

Multiple zeta values (MZVs) generalize the values of the Riemann zeta function at integer points and form a fascinating algebra of real numbers. They appear in a wide variety of contexts, ranging from the theory of associators to the computation of amplitudes in particle physics. Since MZVs are periods, it is natural to introduce their motivic versions, which are acted upon by a motivic Galois group. Surprisingly enough, the Galois theory of motivic MZVs can be made entirely explicit and used to prove powerful theorems on real MZVs. The goal of this minicourse will be to explain the proofs of these theorems, with a special emphasis on Brown's recent proof of a conjecture of Hoffman. The relevant motivic framework is that of mixed Tate motives and their tannakian formalism, which we will review.

PETER JOSSEN – *Exponential motives and exponential periods*

In my lectures, I will present joint work with Javier Fresán. Our departing point is the observation that several interesting transcendence theorems and conjectures are about numbers which presumably are not periods in the usual sense, so Grothendieck's period conjecture says nothing about them. One such case is the Lindemann-Weierstrass theorem which implies for example that e and $e^{\sqrt{2}}$ are algebraically independent, another one is the Rohrlich-Lang conjecture which claims that for any integer $n \geq 3$, the transcendence degree of the field

$$\mathbb{Q}(\Gamma(\tfrac{1}{n}), \Gamma(\tfrac{2}{n}), \Gamma(\tfrac{3}{n}), \ldots, \Gamma(\tfrac{n-1}{n})) \qquad \text{with} \quad \Gamma(s) := \int_0^\infty x^{s-1}e^{-x}dx$$

is equal to $\frac{1}{2}\varphi(n)+1$. Another rich source of transcendence statements is the Siegel-Shidlovskii theorem, which shows for example that the number

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\cdot n!}=\iint_{0\leq x,y\leq 1}e^{-xy}dxdy$$

is transcendental.

In their celebrated paper on periods, Kontsevich and Zagier mention that it should be possible to enlarge Nori's tannakian category of mixed motives to a tannakian category of *exponential motives*, together with realisation functors and comparison isomorphisms between them. Whereas classical motives are associated to varieties, exponential motives are associated to pairs (X, f), where X is an algebraic variety, and f is a regular function on X. Periods for exponential motives, which we call *exponential periods*, typically look like

$$\int_{\gamma}\omega e^{-f}$$

where γ is a topological cycle on $X(\mathbb{C})$ and ω an algebraic differential form. In particular, all the examples above concern exponential periods, and can be recast in terms of the period conjecture extended to exponential motives.

An optimistic outline for my lectures:

(1) Construct some elementary cohomology theories for pairs (X, f), and then the category of exponential motives as a universal cohomology theory following Nori's method.
(2) Formulate the exponential period conjecture and give some examples and consequences. Compare to the classical period conjecture.
(3) Construct some more involved cohomology theories, in particular the Hodge realisation for exponential motives.
(4) Show on concrete examples how the Hodge realisation helps to compute motivic fundamental groups.

Talks

ISHAI DAN-COHEN – *Progress on rational motivic path spaces*

A central ingredient in Kim's work on integral points of hyperbolic curves is the "unipotent Kummer map" which goes from integral points to certain torsors for the prounipotent completion of the fundamental group, and which, roughly speaking, sends an integral point to the torsor of homotopy classes of paths connecting it to a fixed base-point. In joint work with Tomer Schlank, we introduce a space Ω of "rational motivic loops", and we construct a double factorization of the unipotent Kummer map which may be summarized schematically as

$$\text{points}\to\text{rational motivic points}\to\Omega\text{-torsors}\to\pi_1\text{-torsors}.$$

Our "connectedness theorem" says that any two motivic points are connected by a non-empty torsor. Our "concentration theorem" says that for an affine curve,

Ω is actually equal to π_1. As a corollary, we obtain a factorization of Kim's conjecture into a union of smaller conjectures with a homotopical flavor. With some luck, I'll also be ready to discuss the problem of delooping in this setting.

MARTIN GALLAUER – *Motivic Galois groups in characteristic 0*

I will survey different approaches by various mathematicians to constructing the Galois group for mixed motives over a field of characteristic 0. I will also try to elucidate the relation among these candidates, and explain why everyone interested in periods should care.

TIAGO JARDIM DA FONSECA – *Higher Ramanujan equations and periods of abelian varieties*

The Ramanujan equations are certain algebraic differential equations satisfied by the classical Eisenstein series E_2, E_4, E_6. These equations play a pivotal role in the proof of Nesterenko's celebrated theorem on the algebraic independence of values of Eisenstein series, which gives in particular a lower bound on the transcendence degree of fields of periods of elliptic curves. Motivated by the problem of extending the methods of Nesterenko to other settings, we shall explain how to generalize Ramanujan's equations to higher dimensions via a geometric approach, and how the values of a particular solution of these equations relate with periods of abelian varieties.

NILS MATTHES – *Twisted elliptic multiple zeta values*

We introduce an analog of multiple zeta values, which is naturally associated to an elliptic curve together with a distinguished set of torsion points, the so-called "twisted elliptic multiple zeta values". They generalize elliptic multiple zeta values, which were previously introduced by Brown–Levin and Enriquez, and are closely related to both cyclotomic multiple zeta values and iterated integrals of modular forms for congruence subgroups. In a similar way as mixed Tate motives over Z help to explain structural properties of multiple zeta values, it is hoped that the algebraic structure of twisted elliptic multiple zeta values can likewise be elucidated by a suitable category of mixed (elliptic) motives. This is joint work (partly in progress) with J. Broedel, M. Gonzalez, G. Richter, O. Schlotterer and F. Zerbini.

ERIK PANZER – *The Galois coaction on ϕ^4 periods*

We discuss the structure of ϕ^4 periods, focussing on the possibility that primitive ϕ^4 periods span a comodule for the motivic coaction. This is joint work with Oliver Schnetz and rests on a recently updated database of hundreds of exact results for primitive graphs with up to eleven loops.

SINAN ÜNVER – *Iterated sum series and p-adic multiple zeta values*

p-adic multi-zeta values are the p-adic periods of the unipotent fundamental group of the thrice punctured line. They turn out to give all the p-adic periods of mixed Tate motives over Z. In this talk, I will give an explicit series representation of these values in all depths. The new tool is a certain regularization trick for p-adic series.

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28.08.2017

J. Ayoub – Triangulated categories of motives and the Kontsevich-Zagier conjecture.

(I) Constructions of cat. of motives

(II) Motives of (rigid) analytic varieties

(III) Motivic Galois groups and periods

(IV) Proof of geom. (relative) version of the Kontsevich Zagier conjecture

§ Intro

Idea behind these constructions, due to Morel-Voevodsky.

In classical topology (homotopy theory), we have these categories:

$$\begin{tikzcd} H \arrow[r, bend left, "\Sigma^\infty"] & SH \arrow[l, bend left, "\Omega^\infty"] \arrow[r, bend left, "\text{Eilenber-Mc Lane}"] & DA \arrow[l, bend left, "H_*"] \arrow[r] & Ab \end{tikzcd}$$

- H: homotopy categ. of top. spaces ⇝ motives
- SH: stable homotopy cat. ⇝ spectra
- DA: derived cat. of ab. groups ⇝ complexes of —
- Ab: cat. of ab gp's.

Morel-Voevodsky idea: there should be an analogous picture for alg. varieties (fix a base field k, fin. schemes $/k$):

$H(k)$ $\quad SH(k)$ $\quad DA(k)$ $\quad Ab(k)$ ⇜ analog categories

and moreover, this should be the category of motives à la Grothendieck

Construction still missing, there is a candidate (c.f. Nori)

Rem. There are still problems in the analogy. E.g.: π_i's in $H(k)$ tend to be trivial

§ Construction

The starting point is to find a context (cat. large enough) where we can speak about varieties and homotopy types ⇝ simplicial sets (i.e. a functor $\Delta^{op} \to Sets$, $\Delta = \{\underline{n} = \{0 < \dots < n\}\}$)

①

First guess: Δ^{op} Sch = category of simp. schemes.

Better guess: Δ^{op} PS = cat. of simplicial presheaves ~~on~~ schemes of sets.

For a variety X, look at the simplicially constant presheaf of sets repr. by X.

$S_\bullet$, $(S_\bullet)_{cst}$, $\underset{\underset{Sch.}{\uparrow}}{\mathcal{U}} \rightsquigarrow (S_\bullet)_{cst}(\mathcal{U}) = S_\bullet$

Rem.
To get $SH(k)$ (resp. $DA(k)$), one changes slightly the context: replace "Δ^{op} Set" by "Spect" or "complex of Ab. gps".

To get a better mix, one inverts formally some arrows.
To do this, we use: • topology (~~in sense~~ Zariski, Nisnevich, étale, ...)
• $\mathbb{A}^n$ contractible

For $X \in Sch$, $Y_\bullet \to X$ hypercover, i.e. $Y_0 \to X$, $Y_1 \to Y_0$ covers in your top.
$\vdots$

$\mathbb{A}^1$

§ Repeat, with more details in the linear variant

Λ a comm. ring

Sm/k : smooth varieties + étale top.

We want to contract the affine line $\mathbb{A}^1$.
We consider $Cpl(PSh(Sm/k; \Lambda)) = Psh(Sm/k; Cpl(\Lambda))$

complexes of presheaves on Sm/k with value in Λ

- If $K_\bullet$ a complex of Λ-mod. $\rightsquigarrow (K_\bullet)_{cst}$
- If $X \in Sm/k$, $X \otimes \Lambda$, given by $X \otimes \Lambda(\mathcal{U}) := hom(\mathcal{U}, X) \otimes \Lambda$

We define a class of $(\mathbb{A}^1, ét)$-local equiv. generated by
• $K_\bullet \longrightarrow L_\bullet$ is an étale-local equiv. if when looking at hom. sheaves, after sheafifying we get an isom. $H_i(K_\bullet) \to H_i(L_\bullet)$

Def. $DA^{\text{eff}, \text{ét}}(k;\Lambda) := Ho_{\mathbb{A}^1, \text{ét}}(Cpl(PSh(Sm/k;\Lambda)))$

Rem. 1) this is a triangulated category

2) If you replace $Cpl(\Lambda)$ by Spt, you get $SH^{\text{eff}, \text{ét}}(k)$.

3) Objects here are kinda boring. But still, the homomorphisms are interesting, because inverting arrows we don't change objects, but maps.

(One can reverse this)

Interesting class of objects:

Def A complex G of presheaves on Sm/k is said to be $(\mathbb{A}^1, \text{ét})$-fibrant if,

(1) $\forall X \in Sm/k$, $H^i(G(X)) \to H^i_{\text{ét}}(X; G|_{Et/X})$ is an isom.

(i.e. kind of an inj. object)

(2) $\forall X \in Sm/k$, $\xrightarrow{\sim} H^i(G(\mathbb{A}^1_X))$

$\Leftrightarrow$ $G \longrightarrow$

Lemma $\forall F$ complex of psh on Sm/k, $\exists F \to G$ $(\mathbb{A}^1, \text{ét})$-equiv. s.t. G is $(\mathbb{A}^1, \text{ét})$-fibrant (even more, one

Lem. If F, G c.p. on Sm/k, G' an $(\mathbb{A}^1, \text{ét})$-f. rep,

$$\mathrm{Hom}_{DA^{\text{eff}, \text{ét}}(k,\Lambda)}(F, G) = \mathrm{Hom}_{D(PSh(Sm/k;\Lambda))}(F, G')$$

E.g. $F = X \otimes \Lambda$

Def. The motive of X is $X \otimes \Lambda$ viewed in $DA^{\text{eff}, \text{ét}}(k;\Lambda)$

$(X \otimes \Lambda)_{(\mathbb{A}^1, \text{ét})\text{-fib}}$

We denote it $M^{\text{eff}}(X)$.

§ Complements ($\mathbb{Q} \subset k$)

Def. $Y \subset X$, $M^{\text{eff}}(X, Y) = \text{Cone}\left(M^{\text{eff}}(Y) \to M^{\text{eff}}(X)\right)$

Def (of motivic cohom.)

$n \geq 0$, $\Lambda(n) := M^{\text{eff}}(\mathbb{P}^1, \infty)^{\otimes n}[-2n]$

$$= M^{\text{eff}}(\mathbb{G}_m, 1)^{\otimes n}[-n]$$

$$H_L^{p}(X, \Lambda(q)) = \text{Hom}_{DA^{\text{eff}, \text{ét}}(k;\Lambda)}\left(M^{\text{eff}}(X), \Lambda(q)[p]\right)$$

étale motivic coh.

Rem $\mathbb{Q} \subset \Lambda$, $H_L^p = H^p_{\substack{\text{Mot} \\ \text{Nisnevich}}}$

Thm (Voevodsky) $\mathbb{Q} \subset \Lambda$ ~~ex~~

$H^p(X; \Lambda(q))$ coincides with Bloch's higher Chow group.

Cor. (Voevodsky + ε) $\mathbb{Q} \subset \Lambda$, $\mathbb{Q} \subset k$.

∃ a fully faithful embedding covariant (giving homological)

$$\text{Chow}^{\text{eff}}(k;\Lambda) \hookrightarrow DA^{\text{eff}, \text{ét}}(k;\Lambda)$$

$$(\mathbb{P}^1, \mathcal{O}) / (\infty, 0) \longmapsto \Lambda(1)[2]$$

§ Stable version

Reminder The category of Chow mot

$M \in \text{Chow}(k;\Lambda)$ is effective if it is a summand of (X, m), $m \geq 0$

Rem. Non-effective Chow motives are important for duality

(X sm. proj. conn., $(X, 0)^\vee = (X, -\dim(X))$)

Q. Possible to extend embedding ~~to~~ of Cor. ~~th~~ to non-eff.?

A. Yes, if you invert tensor product by $\Lambda(1)$.

Best way: do this first before inverting $(\mathbb{A}^1, \text{ét})$-local equiv

More precisely, we consider the cat. of T-spectra on Sm/k.

T-Spectra $\longrightarrow$ { A T-spectrum is a collection $(E_n)_{n \in \mathbb{N}}$

E_n presh. on Sm/k + bounded maps $E_n \to E_{n+1}$

They form a category $Spt_T(PSh(Sm/k; \Lambda))$.

Ex 1) F a compl. of presh. on Sm/k.

$$\Sigma^\infty_T F = \{T^{\otimes n} \otimes F\}_{n \in \mathbb{N}}$$

2) $$Sus^p_T(F) = \{T^{\otimes n-p} \otimes F\}_{n \in \mathbb{N}}$$

Def. A spectrum $E = (E_n)_{n \in \mathbb{N}}$ is said to be stably $(\mathbb{A}^1, ét)$-fibrant if:

1) Each E_n is $(\mathbb{A}^1, ét)$-fibrant $\left\{ \begin{array}{l} E_n \text{ inj. res.} \\ \text{invariance wrt } \mathbb{A}^1 \end{array} \right.$

2) For the bounded map $E_n \to E_{n+1}$, we ask

$$E_n \xrightarrow{\gamma_n'} \underline{Hom}(\mathbb{G}_m, E_{n+1})$$

Def $DA^{ét}(k; \Lambda) := Ho_{(\mathbb{A}^1, ét)\text{-}st}(Spect \_\_)$

Thm (Voevodsky + ε) (char $k = 0$)

1) $DA^{eff, ét}(k, \Lambda) \hookrightarrow DA^{ét}(k, \Lambda)$

2) $Chow(k, \Lambda) \longrightarrow DA^{ét}(k, \Lambda)$

§ Constructible motives (Geom. motives) $M(X) = \Sigma^\infty_T(X \otimes \Lambda)$

Def. $DA^{ét}_{ct}(k;\Lambda) \subset DA^{ét}(k;\Lambda)$

The sub-Δ-cat stable by direct summand and gen.

by $M(X)(n)$, $X \in Sm/k$, $n \in \mathbb{Z}$

We call them constructible

Thm. $(\mathbb{Q} \subset k)$ Every compact motive is strongly dualisable.

Pf: $Chow(k;\Lambda) \hookrightarrow DA^{ét}(k;\Lambda)$ is a monoidal functor

here they are str. dualizable $\rightsquigarrow$ $M(X)$, X sm. proj. is strongly dualizable

Rem. To get duals, we need $DA^{ét}(k;\Lambda) = Spt_T\, Cpl(PSh(Sm/k;\Lambda))[W^{-1}_{A^1\text{-ét-st}}]$

§ Motives of complex analytic varieties

We replace in the construction:

- Sm/k by $CpVar$ (complex manifolds)
- ét — cl
- A^1 — $\mathbb{D}^1 = \{z \in \mathbb{C};\ |z| < 1\}$
- T — $T^{an} = (\mathbb{G}^{an}_{m,\cdot}) \otimes \Lambda$

The triangulated categories are then $\mathrm{An}\,DA^{(eff)}(\Lambda)$

<u>Rem</u>. If $\varepsilon : k \hookrightarrow \mathbb{C}$, $\quad X \mapsto X^{an} = X(\mathbb{C})$

This induces a functor

$$\mathrm{An}^* : DA^{(eff), \acute{e}t}(k ; \Lambda) \longrightarrow \mathrm{An}\, DA^{(eff)}(\Lambda)$$

$$M(X) \longmapsto M(X^{an})$$

<u>Prop</u>. The obvious functors $D(\Lambda) \xrightarrow{(\cdot)_{cst}} \mathrm{An}\, DA^{eff}(\Lambda) \xrightarrow{\Sigma^{\infty}_{\mathbb{T}^{an}}} \mathrm{An}\, DA(\Lambda)$

are equivalences of categories.

<u>Rem</u>. Everything is locally contractible

<u>Pf</u>. to prove that $(-)_{cst}$ is an equiv. we need to check two properties:

1 - the image of $(-)_{cst}$ generates $\mathrm{An}DA^{eff}(\Lambda)$

2 - the functor is fully faithful

1. It is clear that $\mathrm{An}DA^{eff}(\Lambda)$ is generated by $X \otimes \Lambda$, $X \in Cp\,Var$

$Y_\bullet \to X$ hypercover for classical top. such that

$Y_n = \bigsqcup$ polydiscs

$$Y \otimes \Lambda \xrightarrow{\sim} X \otimes \Lambda \quad \text{is a } cl\text{-local equiv.}$$

$$\downarrow ? \qquad \text{since} \quad \mathbb{D}^n \otimes \Lambda \to pt \otimes \Lambda \ \ (\text{contractible!})$$

$$\underbrace{\pi_0(Y) \otimes \Lambda}_{\in \text{ image}}$$

2. One reduces to the following

$$\mathrm{Hom}_{D(\Lambda)}(\Lambda, \Lambda[i]) \overset{?}{\xrightarrow{\sim}} \mathrm{Hom}_{\mathrm{An}DA^{eff}(\Lambda)}(\Lambda_{cst}, \Lambda_{cst}[i])$$

$$\| \qquad\qquad\qquad\qquad \downarrow$$

$$\begin{cases} \Lambda & \text{if } i = 0 \\ 0 & \text{else} \end{cases}$$

To compute this, we need to find an $(\mathbb{D}^1, cl)$ — fibrant (classical)

~~refinement~~ replacement G of Λ_{cst} the sh...

$\mathrm{Hom}_{D(PSh\,-)}(\Lambda_{cst}, G[i])$

classical local top.

Fact: let $\mathcal{F}$ be a cl-fibrant replacement of Λ_{cst} (i.e. injective resolution of $a_{cl}(\Lambda_{cst})$). Then $\mathcal{F}$ is already a $(\mathbb{D}^1, cl)$-fibr. replacement.

associated sheaf for classical top.

Pf. At the end, one just checks $\mathcal{F}(X) \xrightarrow[\text{quasi-isom.}]{} \mathcal{F}(\mathbb{D}^1 \times X)$

$$H^i(\mathcal{F}(X)) = H^i_{sing}(X) \simeq H^i_{sing}(\mathbb{D}^1 \times X)$$

Then, $RHS = \mathrm{Hom}_{D(PSh)}(\Lambda_{cst}, \mathcal{F}[i]) = H^i(\Gamma(pt, \mathcal{F})) =$

$$= H^i_{sing}(pt; \Lambda) = \begin{cases} \Lambda & \text{if } i=0 \\ 0 & \text{else} \end{cases}$$

[Σ^∞_T equiv. because $M^{eff}(\mathbb{G}^{an}_{m,1})$ is already invertible] $\square$

Def. $k \xhookrightarrow{\sigma} \mathbb{C}$ fixed. Define Betti realisation to be

adjoint given by $(-)_{cst}$

$$B^*_\sigma : DA^{\acute{e}t}(k;\Lambda) \xrightarrow{An^*} \mathrm{An}DA(\Lambda) \simeq D(\Lambda)$$

given by $R\Gamma(pt; -)$

Rem. By construction, B^*_σ has a right adjoint $B_{\sigma,*}$

Rem. $B_{\sigma,*}\Lambda \in DA^{\acute{e}t}(k;\Lambda)$ it represents singular cohomology.

This is related with some path space

§2 b

§ Intro to rigid analytic geom.

Def. If A is a ring, a non-archim. (semi)norm on A is a map $|\cdot| : A \to \mathbb{R}_+$ s.t.

1) $|0| = 0$, $|1| \leq 1$ $\quad (|a| = 0 \Leftrightarrow a = 0)$ $\quad$ i.e. $|1| \in \{0, 1\}$
2) $|ab| \leq |a| \cdot |b|$
3) $|a+b| \leq \max(|a|, |b|)$

$$A^\circ = \{ a \in A \mid |a| \leq 1 \} \qquad \tilde{A} = A^\circ / A^{\circ\circ}$$

$$A^\vee := A^{\circ\circ} = \{ \qquad < \quad \}$$

- $|\cdot|$ multiplicative if $|ab| = |a|\,|b|$.
- A valuation is multi. norm.

Fix k a complete field with a non-trivial valuation

Def (Tate algebra) $k\{t_1, \dots, t_n\}$ = ~~power~~ convergent power series, i.e.

$$\sum a_I t^I, \quad |a_I| \underset{|I| \to \infty}{\longrightarrow} 0$$

with the valuation (Gauss-norm)

$$|f| := \sup |a_I|$$

Rem. $k\{t_1, \dots, t_n\}$ noeth, ideals are closed

- Every morphism between Tate alg. is continuous
- The residue field of maximal ideals are fin. ext. of k.

Def. An affinoid k-algebra A is isom. to a quotient of a Tate algebra, i.e. a k-algebra which admits a surjection from a Tate alg.

Rem $k\{t_1, \dots, t_n\} \twoheadrightarrow A \rightsquigarrow$ an induced norm on A

All these norms are equivalent, they depend on the surjection but define same topology.

<u>Construction</u>: one associates to an affinoid k-alg A

a space $(\mathrm{Spm}(A), \mathcal{O})$

One can glue them to get more general rigid analytic varieties.

$\mathrm{Spm}(A) = \{\text{maximal ideals in } A\}$

<u>Rational domains</u> in $\mathrm{Spm}(A)$ are

$$D(f_0 \mid f_1, \ldots, f_n) = \{a \in \mathrm{Spm}(A) \mid \quad |f_0(a)| \geq |f_i(a)| \}$$

and $(f_0, \ldots, f_n) = A$.

↳ Analog of

$$\begin{array}{c} \simeq \mathrm{Spec}(A_f) \\ D(f) \subset \mathrm{Spec}(A) \\ \shortparallel \\ \{p \not\ni f\} \end{array}$$

<u>Rem</u>. $D(f_0 \mid f_1, \ldots, f_n) \simeq \mathrm{Spm}(B)$

$$B = A\{t_1, \ldots, t_n\} / (f_0 t_i - f_i)$$

This is how we construct the structure sheaves.

<u>Topology</u>. Covers are obtained by taking families of $\mathrm{Spm}(A$

$$D(f_i \mid f_0, \ldots, \widehat{f_i}, \ldots, f_n)_{0 \leq i \leq n}$$

<u>The viewpoint of Raynaud</u>

Idea: if you start with formal scheme of fin. type $\mathfrak{X}$ $= \mathrm{Spf}(A)$ on $\mathrm{Spf}(k^\circ)$.

(the defining ideal contains π, $\forall \pi \in k^{*}$)

Things like $k[[x,y]]$ are forbiden

$\mathfrak{X}_\eta$ Raynaud generic fiber is a rigid analytic variety

k°-alg A $\rightsquigarrow$ $A[\pi^{-1}]$ an affinoid algebra $\rightsquigarrow$

$\rightsquigarrow$ associate maximal spectrum, glue and get $\mathfrak{X}_\eta$.

<u>Rem</u>. $\mathfrak{X}' \longrightarrow \mathfrak{X}$ an admissible blow up (in the sense of formal schemes). Then $\mathfrak{X}'_\eta = \mathfrak{X}_\eta$

~~Moreover,~~

Let X be a rigid analytic variety, quasi-compact, find $\mathfrak{X}$ a formal model (i.e. $X \simeq \mathfrak{X}_\eta$).

Then we can understand coverings:

if $U \subset X$ admissible open, $(U_i)_{i\in I}$ admissible cover of U, then $\exists\ \mathfrak{X}' \to \mathfrak{X}$ admissible blow up, $\mathcal{U}' \subset \mathfrak{X}'$ Zariski open, $(\mathcal{U}'_i)_{i\in I}$ cover of $\mathcal{U}'$ s.t. the generic fibers give us what we want.

<u>Rem</u> For example, we have these opens $D(f_0 \mid f_1, \dots, f_n)$

Let $\mathfrak{X}$ be affine, then the standard open covering of the blow up of some ideal related with gives you these opens.

§ <u>Rigid analytic motives</u>

Repeat construction of $DA^{\text{ét}}$ in the following setting:

- $\mathrm{Sm}\,\mathrm{Rig}/k$, étale topology $\qquad X \longmapsto X^{an}$

- Instead of affine line, take $\qquad \mathrm{Sm}/k \longrightarrow \mathrm{Rig}\,\mathrm{Sm}/k$

  $\mathbb{B}^1 = \mathrm{Spm}(k\{t\})$ to be contracted.

$T^{an} = (\mathbb{G}_m^{an}, 1) \otimes \Lambda \qquad (\simeq (\partial \mathbb{B}^1, 1) \otimes \Lambda).$

$\rightsquigarrow$ we get $\mathrm{RigDA}^{(\mathrm{eff}), \text{ét}}(k;\Lambda)$

$\longrightarrow$ we get a functor by naturality $\mathrm{Rig}^*: \mathrm{DA}^{(\mathrm{eff}), \text{ét}} \longrightarrow$

$\longrightarrow \mathrm{RigDA}^{(\mathrm{eff}, \text{ét})}$ $\quad$ ~~via~~ : $M(X) \mapsto M(X^{an})$

<u>Rem</u>. As in complex analytic setting, we expect some big loss of information
"we will end up with information coming from the special fiber"

Let $k = \tilde{k}((\pi))$, $ch(\tilde{k}) = 0$

<u>Notation</u>: $\mathrm{qu}\,\mathrm{DA}^{(\mathrm{eff}), \text{ét}}(\tilde{k};\Lambda) \subset \mathrm{DA}^{(\mathrm{eff}), \text{ét}}(\mathbb{G}_{m\tilde{k}};\Lambda)$

quasi-unipotent

Triangulated subcategory generated by motives of smooth $\mathbb{G}_{m\tilde{k}}$-schemes of the form:

by direct sums

$\mathrm{DA}^{\text{ét}}(S)$ is obtained by replacing $Sm/k \hookrightarrow Sm/S$

$$X \times \mathbb{G}_{m\tilde{k}} \xrightarrow{pr} \mathbb{G}_{m\tilde{k}} \xrightarrow{(-)^n} \mathbb{G}_{m\tilde{k}}$$

"quasi" comes from here

$n \in \begin{cases} \mathbb{N} & \text{for (-)eff} \\ \mathbb{Z} & \text{else} \end{cases}$

<u>Thm</u>. The composition

$$\begin{array}{ccc} \mathrm{qu}\,\mathrm{DA}^{\text{ét}}(\tilde{k};\Lambda) & & \\ \big\downarrow & & \\ \mathrm{DA}^{\text{ét}}(\mathbb{G}_{m\tilde{k}};\Lambda) & \xrightarrow{\pi^*} & \mathrm{DA}^{\text{ét}}(k;\Lambda) \\ & & \big\downarrow \mathrm{Rig}^* \\ & & \mathrm{RigDA}^{\text{ét}}(k;\Lambda) \end{array}$$

π uniformizer of $k = \tilde{k}((\pi))$

is an equiv. of categories

Rem. 1) Something about monodromy operator

2) ~~Ana~~ This is the analog of the equiv.

$$D(\Lambda) \xrightarrow{\sim} \mathrm{An}\, DA^{ét}(\Lambda),$$

so we think on quasi-unipotent as derived cat. of Λ-modules

Sketch

Step 1. Image of functor generates everything $\rightsquigarrow$ "easier"

Step 2 Fully faithful. $\to$ "harder". Similar, but one needs to understand how to make something fibrant

(A) uses Raynaud's point of view: Assume locally

$X = \mathfrak{X}_\eta$, $\mathfrak{X}$ semi-stable reduction (with multiplicities)

we can do this because we are in equal characteristic $= 0$

so we can resolve singularities of special fiber.

Argue by induction on # branches:

1 branch: $\Rightarrow$ $\mathfrak{X}$ is smooth,

$M(\mathfrak{X}_\eta) =$ image of $M(\mathfrak{X}_\sigma \times \mathbb{G}_m \to \mathbb{G}_m)$

C case, $(\mathfrak{X} \setminus C)_\eta \to (\mathfrak{X})_\eta$

§ Nearby motives

What is the analog of the Betti realisation? We had

$$DA^{ét} \to \mathrm{An}\, DA^{ét} = D(\Lambda)$$

Now

$$\underbrace{DA^{ét}(k) \xrightarrow{Rig^*} Rig\, DA^{ét}(k) \simeq qu\, DA^{ét}(\tilde{k}) \xrightarrow{1^*} DA^{ét}(\tilde{k})}_{\Psi}$$

This is the nearby cycle $\to \Psi$

Rem - In unequal characteristic there are some statements:

$$\mathrm{Rig}\,DA^{\acute{e}t}(k)_{g.r.}$$

$$\simeq$$

$$u\,DA^{\acute{e}t}(\tilde{k})$$

good reduction

unipotent, i.e. you only look at $X \times \mathbb{G}_m \to \mathbb{G}_m$ and generate from this.

Q. How to see logarith = monodromy operator from here?

$$\mathrm{Spec}(1^* 1_* \Lambda(0)) = \hat{\mathbb{Z}} \times \mathbb{G}_a$$

Goal <u>Introduce the motivic Galois gp of a field $k \hookrightarrow \mathbb{C}$</u>

$B_\sigma^* : DA^{\acute{e}t}(k;\Lambda) \to D(\Lambda)$ will play the role of a fiber functor.

§ <u>A weak version of the Tannakian formalism</u>

$f : \mathcal{M} \to \mathcal{E}$ a symmetric monoidal functor.

Q. Is it possible to enrich f in a <u>universally</u> way as follows?

$$\begin{tikzcd} \mathcal{M} \arrow[r, "\tilde{f}"] \arrow[dr, "f"'] & \mathrm{coMod}_{\mathcal{E}}(H) \arrow[d, "\mathrm{forg.}"] \\ & \mathcal{E} \end{tikzcd}$$

$\in \mathcal{E}$ a Hopf algebra or bi-algebra.

<u>Rem</u>. If $\mathcal{M}$ is a tannakian category, f fiber functor, the question has a positive answer $\rightsquigarrow H = \mathcal{O}(\underline{\mathrm{Aut}}^{\otimes}(\omega))$

How to guess what is H?

$\forall M \in \mathcal{M}$, we have $f(M) \xrightarrow{a} f(M) \otimes H$

If $a : H \longrightarrow 1\!\!1 \in \mathcal{E}$, $\rightsquigarrow$ a natural transformation

$f(M) \to f(M)$

$$f(M) \longrightarrow H \otimes f(M) \xrightarrow{a \otimes \mathrm{Id}} f(M)$$

Assume that f has a right adjoint g:

$$(f \to f) \iff (f \circ g \to \mathrm{Id}_{\mathcal{E}})$$

Evaluat $1\!\!1 \rightsquigarrow f \circ g\, 1\!\!1 \longrightarrow 1\!\!1$

<u>Guess</u>: $H = fg\,1\!\!1$: we get back a from $fg\,1\!\!1 \to 1\!\!1$

$\hookrightarrow$ This is correct.

<u>Hypothesis</u>

(1) Assume that f has a right adjoint

(2) f has a monoidal section $e : \mathcal{E} \to \mathcal{M}$

(3 weak) $\forall A, B \in \mathcal{E}$ we want that $gA \otimes e(B) \to g(A \otimes fe(B))$ is an isom.

$$g(A \otimes fe(B)) \overset{e \text{ is a section}}{\cong} g(A \otimes B)$$

(3 strong) $\forall A \in \mathcal{E}$, $B' \in \mathcal{M}$, $gA \otimes B' \to g(A \otimes f(B'))$ is an isom.

(+ e has a right adjoint).

<u>Prop</u>. a) Under (1), (2), (3 weak), $H = fg\,1\!\!1$ is a <u>bialgebra</u> in $\mathcal{E}$ and we have a universal enrichment

b) under (1), (2), (3 strong), H is a Hopf algebra.

<u>Pf</u>. ( ).

<u>Prop</u>. (1), (2), (3 strong) hold for Betti realization $B^*_\sigma : DA^{\acute{e}t}(k;\Lambda) \to D(\Lambda)$

Rem. (3 weak) holds for effective motives.

$\rightsquigarrow \mathcal{H}^{(\mathrm{eff})}_{\mathrm{mot}}(k,\sigma;\Lambda)$ these are the motivic (bi- or Hopf) algebras

$$\mathcal{H}_{\mathrm{mot}} \simeq \mathcal{H}^{\mathrm{eff}}_{\mathrm{mot}}[\tau^{-1}], \quad \tau \in H^0(\mathcal{H}^{\mathrm{eff}}_{\mathrm{mot}})$$

$$B^*_\sigma(\Lambda(1)) \xrightarrow[a]{\sim} \Lambda \quad \Rightarrow \quad \Lambda(1) \xrightarrow{a'} B_{\sigma,*}\Lambda$$

$$\begin{array}{c} \wr \\ B^*_\sigma(\quad) \end{array} \quad \text{and}$$

precompose with a^{-1} $\rightsquigarrow$ $\Lambda \longrightarrow B^*_\sigma B_{\sigma,*}\Lambda : 1 \longmapsto \tau$

Rem. - $\mathcal{H}^{(\mathrm{eff})}_{\mathrm{mot}}(k,\sigma;\Lambda) = \mathcal{H}^{(\mathrm{eff})}_{\mathrm{mot}}(k,\sigma;\mathbb{Z}) \otimes \Lambda$

- $\mathcal{H}^{(\mathrm{eff})}_{\mathrm{mot}}(k,\sigma;\mathbb{Z}/p^n\mathbb{Z}) \simeq C^0(\mathrm{Gal}(\bar{k}/k);\mathbb{Z}/p^n\mathbb{Z})$

Rem. $M \in DA^{\acute{e}t}(k;\Lambda)$

$B^*_\sigma(M)$ comes equipped with a coaction of $\mathcal{H}_{\mathrm{mot}}(k,\sigma)$, this structure determining many things (Hodge structures Galois action),

One can do slightly better, from the pt of view of homotopical algebra.

$\exists$ a finer construction $\rightarrow \underline{\mathcal{H}_{\mathrm{mot}}}(k,\sigma)$, an enhanced version.

Δ-ted functor: $DA^{\acute{e}t}(k,\mathbb{Q}) \longrightarrow h_0\,\mathrm{coMod}(\underline{\mathcal{H}_{\mathrm{mot}}}(k,\sigma))$

Conj: this induces an equivalence on compact objects.

Conj $\mathcal{H}_{\mathrm{mot}}(k,\mathbb{Q}^\sigma)$ has no cohom. except in degree zero.

$\rightsquigarrow \mathcal{O}(G_{\mathrm{mot}}(k,\sigma))$

$\Rightarrow DA^{\acute{e}t}_{ct}(k,\mathbb{Q}) \simeq D^b(\mathrm{Rep}(G_{\mathrm{mot}}(k;\sigma)))$

$\wr$ Nori's motivic Galois gp

§ (-1) connectivity.

Thm $\mathcal{H}^{(eff)}_{mot}(k,\sigma;\mathbb{Q})$ is (-1)-connected $H_i(-)=0$ if $i<0$

Def. $G_{mot}(k;\sigma) = \mathrm{Spec}\, H_0(\mathcal{H}_{mot}(k;\sigma))$

– $B_\sigma^* B_{\sigma,*}\mathbb{Q}$ we want to "compute". We need:

· good model of B_σ^* (how to compute sing. hom. of a motive)

· good model of $B_{\sigma,*}\mathbb{Q}$

Prop. (Groth. comp. isom.)

$$(B_{\sigma,*}\mathbb{Q})\otimes\mathbb{C} \simeq (\Omega_{/k})\otimes_k \mathbb{C}$$

alg. of de Rham complex, nice $\quad D(0,r)^n_{r>1}$

Notation: · $\overline{\mathbb{D}}^n = \{(z_1,\dots,z_n) \mid |z_i|\le 1\}$, as a pro-analytic variety.

· $\overline{\mathbb{D}}^n_{ét}$, a pro-scheme "approximating" $\overline{\mathbb{D}}^n$.

It is indexed by pairs $(\mathcal{U}, i)$, where $\mathcal{U}$ is an étale $\mathbb{A}^n_k$-scheme, and $i: \overline{\mathbb{D}}^n \longrightarrow \mathcal{U}^{an}$

$$\mathcal{U}^{an} \to \mathbb{C}^n$$

$\overline{\mathbb{D}}^n_{ét}: (\mathcal{U}, i) \mapsto \mathcal{U}$.

Rem. $\overline{\mathbb{D}}^n_{ét}$ is pro-étale over $\mathbb{A}^n_k$ affine.

$$\mathcal{O}(\overline{\mathbb{D}}^n_{ét}) = \mathcal{O}(\overline{\mathbb{D}}^n) \cap \overline{k(z_1,\dots,z_n)}^{alg}.$$

Rem. $\overline{\mathbb{D}}^\bullet$ get a cocubical object.

Prop. If $\mathcal{F}$ is a complex of psh. on Sm/k,

$$B_\sigma^*(\mathcal{F}) \simeq \mathrm{Tot}(\mathcal{F}(\overline{\mathbb{D}}^\bullet_{ét}))$$

§ Formal periods (aka motivic periods)

$\sigma : k \hookrightarrow \mathbb{C}$

$B_\sigma^* : DA^{ét}(k; \mathbb{Q}) \to D(\mathbb{Q})$

Yesterday: $B_\sigma^*(\underline{\Omega}/k) = \mathcal{P}(k,\sigma)$, effective version $\mathcal{P}^{eff}(k,\sigma)$

$\underline{\Omega}/k = \{ \Omega^\bullet_{/k}[n] \}_{n \in \mathbb{N}}$

<u>Def.</u> $\mathbb{P}^{(eff)}(k,\sigma) := H_0(\mathcal{P}^{(eff)}(k,\sigma))$ $\rightsquigarrow$ algebra of formal periods

<u>Rem.</u> 1) $\underline{2\pi i} \in \mathbb{P}^{eff}(k,\sigma)$ s.t. $\mathbb{P}(k,\sigma) \simeq \mathbb{P}^{eff}(k,\sigma)[\underline{2\pi i}^{-1}]$

2) Let A be a k-alg.

$\mathbb{P}(k,\sigma) \longrightarrow A \iff$ comparison isom

$B_\sigma^*(\underline{\Omega}/k) \longrightarrow A$ by adj. the same

$\underline{\Omega}/k \to (B_{\sigma,*}\mathbb{Q}) \otimes A$

repr. de Rham cohom   repr. sing. cohom.

3) $\mathcal{P}^{eff}(k,\sigma)$ is "computable", i.e. one can write down a nice complex quasi-isom to it.

$B_\sigma^* F = \mathrm{Tot}(F(\overline{\mathbb{D}}_{ét}))$ a co-cubical pro-sch.

$\mathcal{P}^{eff}(k,\sigma) \simeq \mathrm{Tot}(\Omega^\bullet_{/k}(\overline{\mathbb{D}}_{ét}))$

By some easy manipulation, one gets:

$$\begin{array}{ccccc} & & & 0 & -1 \\ \widetilde{\Omega}^{\infty-n}_{alg}(\overline{\mathbb{D}}^\infty) & \xrightarrow{d} & \cdots & \to \widetilde{\Omega}^{\infty}_{alg}(\overline{\mathbb{D}}^\infty) & 0 \end{array}$$

where $\overline{\mathbb{D}}^\infty$ = pro-analytic variety $\{\overline{\mathbb{D}}^n\}_{n\in\mathbb{N}}$, $\overline{\mathbb{D}}^{n+1} \to \overline{\mathbb{D}}^n : (z_1,\dots,z_{n+1}) \mapsto (z_1,\dots,z_n)$

$\mathcal{O}(\overline{\mathbb{D}}^\infty) = \operatorname{colim}_n \mathcal{O}(\overline{\mathbb{D}}^n) = \{ f(z_1,\dots,z_m,\dots)$ depending only of fin. many of the z_i's $\}$

$$\Omega^{\infty-d}(\bar{\mathbb{D}}^\infty) = \operatorname{colim}_n \Omega^{n-d}(\bar{\mathbb{D}}^n)$$

$$\underbrace{w\, dz_1 \wedge \cdots \wedge \widehat{dz_n} \wedge \cdots \wedge}_{\text{we remove the } d \text{ terms}}$$

$$\Omega^{n-d}(\bar{\mathbb{D}}^n) \to \Omega^{n+1-d}(\bar{\mathbb{D}}^{n+1})$$
$$w \longmapsto w \wedge dz_{n+1}$$

$\tilde{\Omega}^{\infty-d} \subset \Omega^{\infty-d}$ consists of diff that vanish if we substitute $\underset{\text{any}}{z_i}$ by 0 or 1.

<u>Cor.</u> $\mathbb{P}^{\text{eff}}(k,\sigma)$ is the k-v.s. $\mathcal{O}_{alg}(\bar{\mathbb{D}}^\infty)$ modulo relations of the form: $\frac{\partial F}{\partial z_i} - F|_{z_i=1} + F|_{z_i=0}$

$F \in \mathcal{O}_{alg}(\bar{\mathbb{D}}^\infty)$, $i \geq 1$

We get also an evaluation map $\int \mathbb{P}(k,\sigma) \longrightarrow \mathbb{C}$

$$[f] \longmapsto \int_{[0,1]^\infty} f$$

The image of $\int$ is the <u>algebra of true periods</u>.

<u>Fact</u>: $\int$ has another more abstract def'n.:

$$B_\sigma^*(\Omega_{/k}) \longrightarrow B_\sigma^*(\Omega_{/k} \otimes_k \mathbb{C}) \simeq B_\sigma^*(B_{\sigma,*}\mathbb{C}) \xrightarrow{\delta} \mathbb{C}$$

If $M \in DA^{ét}(k;\mathbb{Q})$ compact,

$w \in H^0_{dR}(M) \Leftrightarrow w: M \to \Omega/k$

$B_\sigma^*(M)$

$\gamma \uparrow$

$\mathbb{Q}$

<u>Conj.</u> (Kontsevich - Zagier). If k is a number field, $\int : \mathbb{P}(k,\sigma) \to \mathbb{C}$ is injective.

Rem. We know the 2 conj. are equivalent.

§ Analogues of this for rigid analytic varieties

Idea: Replace discs $\mathbb{D}^n$ by the Tate rigid analytic balls $\mathbb{B}^n$.

Let F be a field with a discrete valuation (role of # field),

K the completion of F, K°, $K^\vee$, $\tilde{K} = K^\circ/_{K^\vee} =: k$

We may define a cocubical pro-scheme $B^\bullet_{\acute{e}t}$, where

$\mathbb{B}^n_{\acute{e}t}$ is the system of étale neigh. in $\mathbb{A}^n_F$ of $\mathbb{B}^n_K$

$\Omega_{/F}(B^\bullet_{\acute{e}t})$ complex of F-v.s.

It can be computed from a simpl. complex

$\Omega^{\infty-\bullet}_{alg}(\mathbb{B}^\infty)$

Def. $P_{rig}(F, v) :=$ H_0 of this complex. $(\Omega^\bullet_{/F}(B^\bullet_{\acute{e}t}))$

Lem. $P_{rig}^{alg}(F, v)$ is the quotient of the F-v.s. $\mathcal{O}_{alg}(\mathbb{B}^\infty)$ by

$$\frac{\partial G}{\partial z_i} - G|_{z_i=1} + G|_{z_i=0}$$

Cor. $\exists$ an evaluation $\int : P_{rig}(F, v) \to K$

$$[f] \mapsto \sum_{I \subset [1,n]} (-1)^{\# I} \tilde{f}(\varepsilon_{I,1}, \dots, \varepsilon_{I,n})$$

$\tilde{f}$ a prim. of f for all the variables

$$\varepsilon_{I,i} = \begin{cases} 0 & i \in I \\ 1 & i \notin I \end{cases}$$

Q. When is $\int : P_{rig}(F, v) \to K$ inj. ?

It is reasonable to expect this in the following cases:

1) $F = \mathbb{Q}$, $K = \mathbb{Q}_p$

2) $F = k(\varpi)$, $K = k((\varpi))$, char $k = 0$. $\to$ not known

Why are these conj. difficult?

If you don't have algebraicity condition, statement is easy.

§ A variant where the question has a positive answer

$F = k(\varpi)$, $K = k((\varpi))$

$\mathbb{B}^n$, $\partial\mathbb{B}^m$. $\quad \mathcal{O}_{alg}(\partial\mathbb{B}^m \times \mathbb{B}^n)$

$\partial\mathbb{B}^m = \mathrm{Spm}(K^\circ[t_i^{\pm 1}] \xrightarrow{\text{completion}} \varpi\text{-}adic)$

Def. $\mathbb{P}_{rig}(F,v)$ the quotient of $\mathcal{O}_{alg}(\partial\mathbb{B}^\infty \times \mathbb{B}^\infty)$ by elements:

1) $\dfrac{\partial G}{\partial z_i} - G|_{z_i=1} + G|_{z_i=0}$

2) $t_j \cdot \dfrac{\partial H}{\partial t_j}$

Fact. $\int : \mathbb{P}_{rig}(F,v) \longrightarrow K$, $\quad \varphi \in \mathcal{O}_{alg}(\partial\mathbb{B}^m \times \mathbb{B}^n)$

$\varphi = \sum\limits_{v \gg -\infty} f_v \cdot \varpi^v$, $\quad f_v \in \mathcal{O}(\mathbb{G}_m^m \times \mathbb{A}^n)$

$\int \varphi = \sum\limits_{v \gg -\infty} \left(\int f_v\right) \cdot \varpi^v$, $\quad \int f = \dfrac{1}{(2\pi i)^m} \oint^{(m)} \int_{[0,1]^n}^{n} f \cdot \dfrac{dt_1}{t_1} \cdots \dfrac{dt_m}{t_m}\, dz_1 \cdots dz_n$

Thm. $\int : \mathbb{P}_{rig}(F,v) \longrightarrow K$ is inj.

§ Sketch of proof

(..)

MOTIVES FOR PERIODS

28.08.17

C. Dupont - Mixed Tate motives and multiple zeta values (MZV)

Reference: Burgos - Fresán (+ Kühn) From numbers to motives

Here: no mention to motivic fundamental groups

§1. The algebra of MZV

$$\zeta(n_1, \dots, n_r) = \sum_{1 \le k_1 < \dots < k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}}, \qquad n_1, \dots, n_{r-1} \geq 1, \quad n_r \geq 2$$

Weight: $n_1 + \dots + n_r$

Depth: r

Depth 1: $\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^n}, \quad n \geq 2$

Euler (1734): $\zeta(2n) = \frac{(-1)^{n-1} B_{2n}}{\cancel{2(2n)}!} (2\pi)^{2n}$, $\quad \sum B_n \frac{t^n}{n!} = \frac{t}{e^t - 1}$

(Bernoulli numbers)

Conjecture: The odd zeta values $\zeta(2n+1)$ are alg. independent over $\mathbb{Q}(\pi)$.

Apéry '79: $\zeta(3) \notin \mathbb{Q}$

Ball - Rivoal '99: $\exists \infty$ many $\zeta(2n+1) \notin \mathbb{Q}$.

Open: $\zeta(5) \in \mathbb{Q}$? (for example)

Denote: Z_n = the $\mathbb{Q}$-span of MZVs of weight n $\subset \mathbb{R}$.

Convention: $\underset{\text{empty}}{\zeta(\phi)} = 1$, $\quad Z_0 = \mathbb{Q}$.

$Z = \sum_{n \geq 0} Z_n$ (we will see that it is a $\mathbb{Q}$-subalgebra of $\mathbb{R}$)

Conj (Zagier, '94).

1) Define an integer sequence (d_n) by

$$\begin{cases} d_n = d_{n-2} + d_{n-3} \\ d_0 = 1, \quad d_1 = 0, \quad d_2 = 1 \end{cases}$$

Then $\dim_{\mathbb{Q}} (Z_n) = d_n \quad \forall n$ $\rightsquigarrow$ "dimension conjecture"

2) The spaces Z_n are in direct sum: $Z = \bigoplus_{n \geq 0} Z_n$

Rem. dim. conjecture is only known for $n \leq 4$.

$n=3$: $\zeta(3) \overset{\text{☆}}{=} \zeta(1,2)$

$n=4$: $\zeta(4), \quad \zeta(2,2), \quad \zeta(1,3), \quad \zeta(1,1,2)$ are pairwise collinear.

$n=5$ $\zeta(5), \quad \zeta(1,4), \ldots$ $\rightsquigarrow$ You can prove that they are all pairwise collinear combinations of $\zeta(2,3), \zeta(3,2)$.

Conj: $\zeta(2,3), \zeta(3,2)$ are linearly independent.

Conj (Hoffman, '97). The MZVs $\zeta(n_1, \ldots, n_r)$ with $n_i \in \{2,3\}$ form a basis of Z. ($\Rightarrow$ dim. conjecture)

§ Motivic MZVs and Brown's thm

We will introduce an algebra $\mathcal{P}^{mot}$ = the algebra of (real, eff.) motivic periods of mixed Tate motives over $\mathbb{Z}$, endowed with a morphism of algebras $\mathrm{per}: \mathcal{P}^{mot} \longrightarrow \mathbb{R}$ "period map"

We will construct elements $\zeta^{mot}(n_1, \ldots, n_r) \in \mathcal{P}^{mot}$ s.t. $\mathrm{per}(-) = \zeta(n_1, \ldots, n_r)$

$Z^{mot} \subset \mathcal{P}^{mot}$ the $\mathbb{Q}$-span of motivic MZVs.

$$\begin{array}{ccc} G \circlearrowright Z^{mot} & \subseteq & \mathcal{P}^{mot} \circlearrowleft G \\ \downarrow & & \downarrow \mathrm{per} \\ \text{MZVs} \rightsquigarrow Z & \longhookrightarrow & \mathbb{R} \end{array}$$

(Groth. $\downarrow$)

So we are replacing numbers by motives.

<u>Conjecture</u> (Grothendieck's period conjecture).

per : $\mathcal{P}^{mot} \longrightarrow \mathbb{R}$ is injective.

<u>Cor</u>: $\mathcal{Z}^{mot} \simeq \mathcal{Z}$

<u>Fact</u>: $\mathcal{P}^{mot}$ is graded by weight: $\mathcal{P}^{mot} = \bigoplus_{n \geq 0} \mathcal{P}_n^{mot}$ and

$\zeta^{mot}(n_1, \ldots, n_r) \in \mathcal{P}^{mot}_{n_1 + \cdots + n_r}$

($\Rightarrow$ Part (2) of Zagier's conj. is true for motivic MZVs).

<u>Fact</u> $\dim_{\mathbb{Q}} \mathcal{P}_n^{mot} = d_n$

In particular, $\dim_{\mathbb{Q}} \mathcal{Z}_n \leq d_n$ (Goncharov, Terasoma '00)

<u>Rem</u>. We don't know a proof of which is non-motivic.

<u>Tannakian formalism</u>: $\mathcal{P}^{mot} \circlearrowleft G$ gp scheme / $\mathbb{Q}$ "motivic Galois gp".

Goncharov, Brown: $\mathcal{Z}^{mot}$ stable by G.

This is kinda Galois theory for motivic MZVs.

<u>Ex</u>. $G.\zeta^{mot}(5) = \{\lambda \zeta^{mot}(5) + \alpha, \ (\lambda, \alpha) \in \mathbb{Q}^\times \times \mathbb{Q}\}$ orbit of dim 2

$G.\zeta^{mot}(2,3) = \{\lambda \zeta^{mot}(2,3) + \beta \zeta^{mot}(2) + \gamma, \ (\lambda, \beta, \gamma) \in \mathbb{Q}^\times \times \mathbb{Q} \times \mathbb{Q}\}$.

orbit of dim 3

In particular, they are not linearly dependence.

<u>Thm</u> (Brown, '12) The motivic MZVs $\zeta^{mot}(n_1, \ldots, n_r)$ with $n_i \in \{2,3\}$ are linearly independent. (Hence, a basis of $\mathcal{Z}^{mot} = \mathcal{P}^{mot}$)

Idea: assume non-trivial linear relation, apply well-chosen elem. of G, get contradiction.

Cor. $\mathcal{Z}^{mot}_n = \mathcal{P}^{mod}_n$

Apply per: $\mathcal{Z}^{mot} \to \mathcal{Z}$: get the spanning part of Hoffman's conjecture.

Cor. $\underbrace{\text{Zagier's conjecture} \iff \text{Hoffman's conjecture} \Leftarrow \text{period conj.}}$

$\Downarrow$

Conjecture on odd zeta values.

§ Double shuffle relations

$\to$ shuffle / quasi-shuffle product

One can write a product of two MZVs of weight m and n as a sum of MZVs of weight $m+n$.

Example: $\zeta(m)\cdot\zeta(n) = \sum_{k=1}^{\infty} \frac{1}{k^m} \sum_{l=1}^{\infty} \frac{1}{l^n} = \left(\sum_{1 \le k < l} + \sum_{1 \le k = l} + \sum_{1 \le l < k}\right) \frac{1}{k^m l^n}$

$$= \zeta(m,n) + \zeta(m+n) + \zeta(n,m)$$

$\rightsquigarrow$ Shuffle product:

Def. Iterated integrals $a_i \in \{0,1\}$. $I(0; a_1 \ldots a_n, 1) =$

$$= \int_{0 \le t_1 \le \ldots \le t_n \le 1} \omega_{a_1}(t_1) \cdots \omega_{a_n}(t_n), \quad \text{where } \omega_0(t) = \frac{dt}{t}, \quad \omega_1(t) = \frac{dt}{1-t}$$

Fact. (Kontsevich): MZVs are integrals of algebraic forms.

$$\zeta(n_1, \ldots, n_r) = I(0; 1, \underbrace{0, \ldots, 0}_{n_1 - 1}, 1, \underbrace{0, \ldots, 0}_{n_2 - 1}, \ldots, 1, \underbrace{0, \ldots, 0}_{n_r - 1}; 1)$$

Proof for $\zeta(2)$.

$$I(0, 1, 0; 1) = \iint_{0 \le x \le y \le 1} \frac{dx}{1-x} \frac{dy}{y} = \sum_k \int_0^1 \frac{dy}{y} \int_0^y x^k dx = \ldots = \sum_{k=0}^{\infty} \frac{1}{(k+1)^2} = \zeta(2)$$

1, 2, 3, 4

1 3

2 4

Def. Shuffle $Ш$

$$Ш(r, n-r) = \left\{ \sigma \in S_n \;\middle|\; \begin{array}{l} \sigma(1) < \dots < \sigma(r) \\ \sigma(r+1) < \dots < \sigma(n) \end{array} \right\}$$

Prop. (shuffle formula)

$$I(0; a_1, \dots a_r; 1)\, I(0; a_{r+1}, \dots a_n; 1) = \sum_{\sigma \in Ш(n, n-r)} I(0; a_{\sigma^{-1}(1)} \dots a_{\sigma^{-1}(n)}; 1)$$

Pf. Use Fubini integral over $\Delta^r \times \Delta^{n-r}$

$$\Delta^r \times \Delta^{n-r} = \bigcup_{\sigma \in Ш(n, n-r)} \Delta^n(\sigma) \longrightarrow \{0 \le t_{\sigma^{-1}(1)} \le \dots \le t_{\sigma^{-1}(n)} \le 1\}$$

Ex:
$$\begin{aligned} \zeta(2)\zeta(3) &= I(0; 1, 0; 1) \cdot I(0; 1, 0, 0; 1) \\ &= 6 \cdot I(0; 1, 1, 0, 0, 0; 1) + 3 \cdot I(0; 10100; 1) + \\ &\quad + I(0; 10010; 1) \\ &= 6\zeta(1,4) + 3\zeta(2,3) + \zeta(3,2) \end{aligned}$$

Rem. We have only defined $I(0; w; 1)$ fo w a word in $\{0, 1\}$ starting with 1 and ending with 0.

There is a unique way of extending this to all words w s.t. the shuffle product formula still holds and $I(0; 0; 1) = 0$, $I(0; 1; 1) = 0$

Ex: prove that $I(0; 010; 1) = -2\zeta(3)$

Extended double shuffle relations

Shuffle product = Shuffle product $\Rightarrow$ linear relation among MZVs

Ex:
$$\zeta(2)\zeta(3) = \begin{cases} \zeta(2,3) + \zeta(5) + \zeta(3,2) \\ 6\zeta(1,4) + 3\zeta(2,3) + \zeta(3,2) \end{cases}$$

Fact: you can also do it with "$\zeta(1) \times \zeta(n_1, \dots, n_r)$"

Ex: $\zeta(1)\zeta(2) = \begin{cases} \zeta(1,2) + \zeta(3) + \text{"}\zeta(2,1)\text{"} \\ 2\zeta(1,2) + \text{"}\zeta(2,1)\text{"} \end{cases}$

even if everything is ∞, this still gives you the relation

$$\zeta(1,2) + \zeta(3) = \cancel{2}\zeta(1,2)$$

There are no other relations, i.e.

Conj. The extended double shuffle relations span the space of linear relations among MZVs.

~~Exercise~~: write all ex double ш relations in weight ≤ 5.

Rem 1) This implies 2^{nd} part of Zagier's conj.

2) we don't know if this conj. is consistent with dim. conj. Numerically, up to $d=20$ is consistent, but we don't have a ~~good~~ general argument.

§ 2. Motivic periods

ξ - Cohomology

$X/\mathbb{Q}$ smooth

1) algebraic de Rham cohom. $H^n_{dR}(X) = H^n(X, \Omega^\bullet_{X/\mathbb{Q}})$

2) Betti / singular cohomology: $H^n_B(X) := H^n_{sing}(X(\mathbb{C}), \mathbb{Q})$

3) Comparison isom.

$$H^n_{dR}(X) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow[\sim]{\int} H^n_B(X) \otimes_{\mathbb{Q}} \mathbb{C}$$

$\rightsquigarrow$ period matrix is the matrix of this isom. with a chosen basis.

Variant $Y \subset X$

$H^n(X, Y)$

Naive category of motives

$(V_{dR}, V_B, \mathrm{comp})$

$\mathbb{Q}$ vect. sp. $\qquad$ isom. between $\mathbb{C} \otimes_{\mathbb{Q}} V_{dR}$ and $\mathbb{C} \otimes_{\mathbb{Q}} V_B$.

$\rightsquigarrow$ naive motives $H^n(X, Y)$.

Def. A naive motivic period is a triple
$$(H^n(X,Y), v, \varphi)$$
with $v \in H^n_{dR}(X,Y)$, $\quad \varphi \in H^n_B(X,Y)^\vee$

The corresponding period is $\langle \varphi, \mathrm{comp}(v) \rangle \quad \left(= \int_\varphi v\right)$

Ex. $X = \mathbb{A}^1_{\mathbb{Q}} - \{0\} = \mathbb{G}_m$

$$H^1_{dR}(X) = \mathrm{coker}\left(\mathbb{Q}[t, t^{-1}] \xrightarrow{d} \mathbb{Q}[t, t^{-1}]\,dt \;:\; f(t) \mapsto f'(t)\,dt\right)$$
$$= \mathbb{Q}\left[\tfrac{dt}{t}\right]$$

$$H^1_B(X)^\vee = H_1^{\mathrm{sing}}(\mathbb{C}^\times; \mathbb{Q}) = \mathbb{Q}[\gamma]$$

Period matrix: $2\pi i \quad \left(= \int_\gamma \frac{dt}{t}\right)$

Notation: $\mathbb{Q}(-1) = H^1(\mathbb{A}^1_{\mathbb{Q}} - \{0\})$.

Def. $(2\pi i)^{\mathrm{mot}} := \left(\mathbb{Q}(-1), \left[\tfrac{dt}{t}\right], [\gamma]\right)$

Also define: $\mathbb{Q}(1) := \mathbb{Q}(-1)^\vee$

$\qquad \mathbb{Q}(-k) := \mathbb{Q}(-1)^{\otimes k}$

$\rightsquigarrow \mathbb{Q}(-k)$, $k \in \mathbb{Z}$, with period matrix $\left((2\pi i)^k\right)$

§ Kummer extensions and the motivic logarithm

$\sim \mathbb{G}_m$

Define: $K_a := H^1(\mathbb{A}^1_{\mathbb{Q}} - \{0\}, \{1, a\})$

If $a \in \mathbb{Q}_{>0}$. Kummer motive of parameter a.

de Rham: $K_{a,dR}$ is the cohom. of the total complex

of $\Omega^\bullet(\mathbb{A}^1_{\mathbb{Q}} - \{0\}) \to \Omega^\bullet(\{1\}) \oplus \Omega^\bullet(\{a\})$

$$\begin{array}{ccc} \mathbb{Q}[t, t^{-1}] & \longrightarrow & \mathbb{Q} \oplus \mathbb{Q} \\ f(t) & \longmapsto & (f(1), f(a)) \\ d \downarrow & & \\ \mathbb{Q}[t, t^{-1}]dt & & \end{array}$$

Exercise: $K_{a,dR}$ has a basis consisting of

$$\frac{1}{a-1}dt = \left(\frac{1}{a-1}dt, 0, 0\right)$$

and $\left(\frac{dt}{t}, 0, 0\right)$

Betti: $K^\vee_{a,B} = H_1^{sing}(\mathbb{C}^\times; \{1, a\}; \mathbb{Q})$

basis $[\delta_{1,a}]$ and $[\gamma]$ $\quad \gamma$

$[\delta_{1,a}]$: straight path from 1 to a.

Period matrix

$$\begin{array}{ccc} [\frac{1}{a-1}dt] & [\frac{dt}{t}] & \\ \left(\begin{array}{c} 1 \\ 0 \end{array}\right. & \left.\begin{array}{c} \log a \\ 2\pi i \end{array}\right) & \begin{array}{c} [\delta_{1,a}] \\ [\gamma] \end{array} \end{array}$$

Define

$$\log^{mot}(a) := \left(K_a, \left[\frac{dt}{t}\right], [\delta_{1,a}]\right)$$

Let's compute the long exact seq. in relative cohom.

$$0 \to H^0(\mathbb{A}^1 \setminus \{0\}) \to H^0(\{1,a\}) \to K_a \to H^1(\mathbb{A}^1 - \{0\}) \to 0$$

$$\begin{array}{ccc} \| & \| & \| \\ \mathbb{Q}(0) & \longrightarrow \mathbb{Q}(0) \oplus \mathbb{Q}(0) & \mathbb{Q}(-1) \\ x & \longmapsto (x \oplus x) & \end{array}$$

Non trivial extension of $\mathbb{Q}(k)$'s.

$\Rightarrow$ short exact seq. $0 \to \mathbb{Q}(0) \to K_a \to \mathbb{Q}(-1) \to 0$

This s.e.s. is not split because $\log(a) \notin \mathbb{Q} \oplus \mathbb{Q}.2\pi i$

This is the prototype of a mixed Tate motive: iterated extension of motives $\mathbb{Q}(-k)$ $(k \in \mathbb{Z})$

$$\begin{array}{c} \mathbb{Q}(-k) \\ \updownarrow \\ ((2\pi i)^k) \end{array}$$

Not a mixed Tate motive: $H^1(C)$ C smooth curve, $g(C) > 0$

§ Motivic dilogarithm and motivic $\zeta(2)$

$$Li_2(t) = \sum_{k=1}^{\infty} \frac{t^k}{k^2}, \quad \text{for } |t| < 1.$$

Analytic continuation will be a multi-valued function

$t \in \mathbb{Q},\ 0 < t < 1$

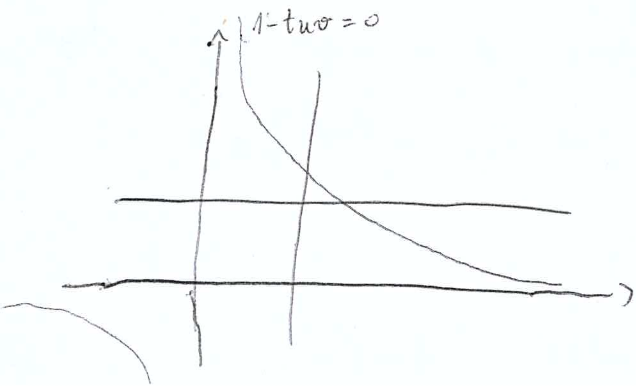
$\to$ blow-up of $(0,0)$

$(x,y) = (uv, v)$

$$\left(Li_2(1) = \zeta(2) \right)$$

Li_2 has an integral representation:

$$Li_2(t) = \iint_{0 \le x \le y \le 1} \frac{t\, dx}{1-tx} \frac{dy}{y} = \int_0^1 \int_0^1 \frac{t\, du\, dv}{1-tuv}$$



$X = \mathbb{A}^2_{\mathbb{Q}} \setminus \{1 - tuv = 0\}$,

$Y = \{u=0\} \cup \{u=1\} \cup \{v=0\} \cup \{v=1\}$

<u>Def</u>. $M_t = H^2(X,Y)$ "dilogarithm motive"

$$M_{t,dR} \ni \left[\frac{t\, du\, dv}{1-tuv}\right]; \quad M^\vee_{t,B} \ni [D]$$

Ex. The period matrix of M_t is

$$\begin{pmatrix} 1 & -\log(1-t) & Li_2(t) \\ 0 & 2\pi i & 2\pi i \log(t) \\ 0 & 0 & (2\pi i)^2 \end{pmatrix}$$

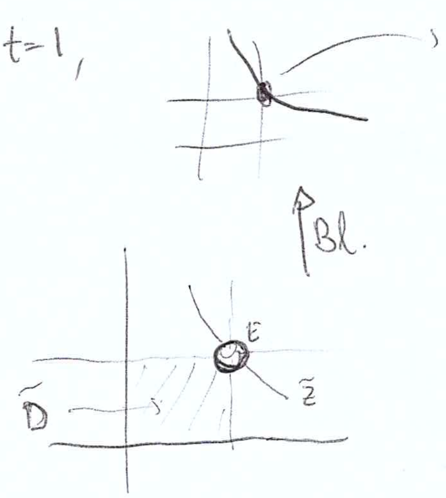
In particular, there are short exact sequences

$$0 \to \mathbb{Q}(0) \to M_t \to K_t(-1) \to 0$$

$$0 \to K_{\frac{1}{1-t}} \to M_t \to \mathbb{Q}(-2) \to 0$$

$\to$ These sequences come from geometry, ~~from~~ they are long exact sequences.

For $t=1$, we blow up the point $(u,v)=(1,1)$



Period matrix.

$$\begin{pmatrix} 1 & \zeta(2) \\ 0 & (2\pi i)^2 \end{pmatrix}$$

~~$H^2(\tilde{X}, \tilde{Y}$~~ $\underbrace{H^2(\tilde{\mathbb{A}}^2 - \tilde{Z}, \tilde{Y} \cup E)}$

$\rightsquigarrow$ Define motivic $Li_2^{mot}(t)$, $\zeta^{mot}(2) = \left(\quad , \left[\frac{du\,dv}{1-uv}\right], [\tilde{D}] \right)$

§ Tannakian formalism

Def. A (neutral, $\mathbb{Q}$-linear) tannakian category is a $\mathbb{Q}$-linear ab. cat. $\mathcal{C}$ equipped with a compatible structure of a rigid tensor category, s.t. $\exists$ exact and faithful functor (fiber) $\omega : \mathcal{C} \to \mathrm{Vect}_{\mathbb{Q}}$

Hom sp. are $\mathbb{Q}$-vec. sp.

Ex. G a gp scheme $/\mathbb{Q}$, $\mathcal{C} = \mathrm{Rep}_{\mathbb{Q}}(G)$

$\omega : \mathrm{Rep}_{\mathbb{Q}}(G) \longrightarrow \mathrm{Vect}_{\mathbb{Q}}$ the forgetful functor

For $\mathcal{C}$ a tannakian category, $\omega : \mathcal{C} \to \mathrm{Vect}_{\mathbb{Q}}$ a fiber functor, define $G_\omega := \underline{\mathrm{Aut}}^{\otimes}(\omega)$ the affine gp scheme of $\otimes$-autom. of ω.

For R a $\mathbb{Q}$-alg., $G_\omega(R)$ = set of automorphisms of the functor $\omega \otimes_{\mathbb{Q}} R : \mathcal{C} \to \mathrm{Mod}_R$, that are compatible with $\otimes$.

__Thm__. (tannakian reconstruction thm). In this setting, we have an equiv. of categories

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\sim} & \mathrm{Rep}_{\mathbb{Q}}(G_\omega) \\ M & \longmapsto & \omega(M) \\ & {\scriptstyle \omega} \searrow \quad \swarrow {\scriptstyle \text{forget}} & \\ & \mathrm{Vect}_{\mathbb{Q}} & \end{array}$$

G_ω = Galois/fundamental gp of $(\mathcal{C}, \omega)$

__Ex__. $\mathcal{C}$ = category of graded vect. sp.

$\omega : \mathcal{C} \to \mathrm{Vect}_{\mathbb{Q}}$ forgets the grading $\quad \mathcal{C} \simeq \mathrm{Rep}(\mathbb{G}_m)$

$\mathbb{G}_m$ acts on $V = \bigoplus_{n \in \mathbb{Z}} V_n$ by the character $t \mapsto t^n$ on V_n

__Rem__. G_ω is usually a very very big gp, and a point there is complicated to find (except the unit).

The natural object is $H_\omega := \mathcal{O}(G_\omega)$ (a Hopf algebra).
It is spanned by triples (M, v, φ) with $M \in \mathrm{Ob}(\mathcal{C})$, $v \in \omega(M)$, $\varphi \in \omega(M)^\vee$.

These triples are called matrix coefficients.

If $\mathcal{C} = \mathrm{Rep}_{\mathbb{Q}}(G)$, (M, v, φ) is a function on G:
$(g \in G_{\mathbb{Q}}) \longmapsto \langle \varphi, g \cdot v \rangle$

These triples are bilinear in v and φ, and satisfy the relations

$\forall f: M \to M'$ morphism in $\mathcal{C}$, $\forall v \in \omega(M)$, $\varphi' \in \omega(M')^\vee$

$$(M', \omega(f)(v), \varphi') = (M, v, \omega(f)^\vee(\varphi'))$$

Exercise: give a formula for the coproduct $\Delta: \mathcal{H}_\omega \to \mathcal{H}_\omega \otimes \mathcal{H}_\omega$

§ Motivic periods

Play with different fiber functors.

Let $\mathcal{C}$ a tannakian category, $\omega_{dR}, \omega_B : \mathcal{C} \to \mathrm{Vect}_{\mathbb{Q}}$ two f.f.

with an isom. $\omega_{dR} \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} \omega_B \otimes_{\mathbb{Q}} \mathbb{C}$ compatible with $\otimes$.

usually $\omega_{dR} \otimes_k \mathbb{C}$, but here $X/\mathbb{Q}$

$\rightsquigarrow$ de Rham Galois gp $G_{dR} = \underline{\mathrm{Aut}}^\otimes(\omega_{dR})$

$\rightsquigarrow$ Betti —"— $G_B = \underline{\mathrm{Aut}}^\otimes(\omega_B)$

Def. $T = \underline{\mathrm{Isom}}^\otimes(\omega_{dR}, \omega_B)$. This is a (G_{dR}, G_B)-bitorsor.

Rem. T very complicated. Instead, we work with:

Def. The algebra of motivic periods of $(\mathcal{C}, \omega_{dR}, \omega_B)$ is the alg. of functions on T: $\mathcal{P}^{mot} := \mathcal{O}(\underline{\mathrm{Isom}}^\otimes(\omega_{dR}, \omega_B))$

$\mathcal{P}^{mot}$ is spanned by triples (M, v, α) with

$M \in \mathcal{C}$, $v \in \omega_{dR}(M)$, $\alpha \in \omega_B(M)^\vee$.

$\left(\int_\alpha f^*(v) = \int_{f_*(\alpha)} v \right)$

$(M', \omega_{dR}(f)(v), \alpha) = (M, v, \omega_B(f)^\vee(\alpha))$

Rem. The comparison isom. is a point in $T(\mathbb{C})$. Evaluating in this point gives us the period map

$$\mathrm{per} : \mathcal{P}^{mot} \longrightarrow \mathbb{C}$$

$$\mathrm{per}(M, v, \alpha) := \langle \alpha, \mathrm{comp}(v) \rangle \qquad \left(= \int_\alpha v \right)$$

<u>Conjecture</u> (Period conjecture for $(\mathcal{C}, \omega_{dR}, \omega_B, \mathrm{comp})$)

$\mathrm{per} : \mathcal{P}^{mot} \longrightarrow \mathbb{C}$ is injective

<u>Rem</u>. 1) In other words, relations between periods are explained by algebraic geometry.

2) How to prove identities between motivic periods?

<u>Ex</u>: $\log^{mot}(ab) = \log^{mot}(a) + \log^{mot}(b)$

$$\log ab = \int_1^{ab} \frac{dt}{t} = \int_1^a \frac{dt}{t} + \int_a^{ab} \frac{dt}{t} \overset{t = au}{=} \int_1^a \frac{dt}{t} + \int_1^b \frac{du}{u} = \log a + \log b$$

• Translate this to a motivic proof!

§ <u>Weight filtration and mixed Hodge-Tate structure</u>

<u>Def</u>. A mixed Hodge-Tate structure is a triple $(H_{dR}, H_B, \mathrm{comp})$ consisting of - a f.d. $\mathbb{Q}$-v.s. H_B together with an increasing filtration

$$\cdots \subseteq W_{2(n-1)} H_B \subseteq W_{2n} H_B \subseteq \cdots$$

- a fin. dim. $\mathbb{Q}$-v.s. H_{dR} with a grading $H_{dR} = \bigoplus_n (H_{dR})_{2n}$

- an isom. $\mathrm{comp} : H_{dR} \otimes_{\mathbb{Q}} \mathbb{C} \longrightarrow H_B \otimes_{\mathbb{Q}} \mathbb{C}$ s.t.

(i) $\forall n$, comp sends $(H_{dR})_{2n} \otimes \mathbb{C}$ to $W_{2n} H_B \otimes_{\mathbb{Q}} \mathbb{C}$

(ii) $\forall n$, comp induces $(H_{dR})_{2n} \otimes \mathbb{C} \simeq (W_{2n} H_B / W_{2(n-1)} H_B) \otimes_{\mathbb{Q}} \mathbb{C}$

sends $(H_{dR})_{2n}$ to $\left(W_{2n}H_B / W_{2(n-1)}H_B\right) \otimes \mathbb{Q}(2\pi i)^n$

<u>Category</u>: MHTS

The period matrix of a MHTS looks like

<u>Rem</u>. This is a M.H structure of Tate type.

<u>Lemma</u> Let (H, W, F) be a mixed Hodge structure such that the Hodge number $h^{p,q} = 0$ for $p \neq q$. Then th,

$$W_{2n} H_{\mathbb{C}} = W_{2(n-1)} H_{\mathbb{C}} \oplus \left(W_{2n} H_{\mathbb{C}} \cap F^n H_{\mathbb{C}}\right)$$

$$H_{dR} = \bigoplus gr^W_{2n} H$$

§3 <u>MT $(\mathbb{Z})$</u>

Fix F a number field

$$\begin{array}{ccc} \overset{\text{Tate type}}{DMT(F)} & \hookrightarrow & DM(F) \\ \uparrow & & \\ MT(F) & & \end{array}$$

$MT(\mathcal{O}_F) \hookrightarrow MT(F)$ abelian category

$MT(\mathcal{O}_F)$: interesting for arithmetic purposes.

$DM(F)$ $\left(= DM_{geom}(F; \mathbb{Q})\right)$ Voevodsky's triangulated category of geometric motives $/F$ with $\mathbb{Q}$-coefficients.

- Special objects of $DM(F)$: complexes of varieties over F

$$\cdots \longrightarrow X^{n-1} \longrightarrow X^n \longrightarrow \cdots$$

$\mathbb{Q}$-linear combination of morphisms (cohomological conventions)

Ex. $\mathbb{Q}(-1)$

$$\underset{\deg -1}{\mathbb{G}_{m,F}} \longrightarrow \underset{\deg 0}{\{1\}} \qquad \left(H^1(\mathbb{G}_{m,F}, \{1\})\right)$$

inclusion of $\{1\}$ in $\mathbb{G}_{m,F}$

we put the $\{1\}$ to get rid of the H^0

Ex. Kummer motives; $a \in F^\times$

$$K_a : \underset{\deg -1}{\mathbb{G}_{m,F}} \longrightarrow \underset{\deg 0}{\{1\} \sqcup \{a\}} \in DM(F)$$

Distinguished triangle in $DM(F)$:

$$\underset{\mathbb{Q}(0)}{(\{a\})} \longrightarrow K_a \longrightarrow \underset{\mathbb{Q}(-1)}{\left(\mathbb{G}_{m,F} \to \{1\}\right)} \xrightarrow{+1}$$

Ex. Define dilogarithm motives $M_t \in DM(F)$, for $t \in F - \{0,1\}$, and prove that we have exact triangles

$$\mathbb{Q}(0) \to M_t \to K_t(-1) \xrightarrow{+1}$$

$$K_{\frac{1}{1-t}} \to M_t \to \mathbb{Q}(-2) \xrightarrow{+1}$$

Thm (Voevodsky, Bloch, Levine, Borel) Let r_1 (resp. r_2) denote the number of real (resp. complex conjugate) embeddings $F \hookrightarrow \mathbb{C}$. Then we have

$$\mathrm{Hom}_{DM(F)}\left(\mathbb{Q}(-n), \mathbb{Q}(0)[p]\right) \simeq \left(K_{2n-p}(F) \otimes_{\mathbb{Z}} \mathbb{Q}\right)^{(n)} \simeq \begin{cases} \mathbb{Q} & p=n=0 \\ F^\times \otimes_{\mathbb{Z}} \mathbb{Q} & p=n=1 \\ \vdots & \end{cases}$$

alg. higher K-theory

$$\mathrm{Hom}_{DM(F)}\big(\mathbb{Q}(-n), \mathbb{Q}(0)[p]\big) \simeq \big(K_{2n-p}(F)\otimes_{\mathbb{Z}}\mathbb{Q}\big)^{(n)} \simeq \begin{cases} \mathbb{Q} & p=0,\ n=0 \\ F^{\times}\otimes_{\mathbb{Z}}\mathbb{Q} & p=1,\ n=1 \\ \mathbb{Q}^{r_2} & \text{for } p=1,\ n>1\ \cancel{\text{odd}}\ \text{even} \\ \mathbb{Q}^{r_1+r_2} & \text{for } p=1,\ n>1\ \cancel{\text{even}}\ \text{odd} \\ 0 & \text{else} \end{cases}$$

Rem. $\mathrm{Hom}_{DM(F)}\big(\mathbb{Q}(-1), \mathbb{Q}(0)[1]\big) = F^{\times}\otimes_{\mathbb{Z}}\mathbb{Q}$

$$[K_a] \longleftrightarrow a$$

so the Kummer extensions are all of those appearing.

Note. This is 0 for $p<0$ ⇝ Beilinson - Soulé vanishing

Def. The triangulated category of mixed Tate motives over F is the full triang. subcategory $DMT(F) \subseteq DM(F)$ generated by the objects $\mathbb{Q}(-n)$, for $n \in \mathbb{Z}$.

Kummer motives: $(a \in F^{\times})$: $\mathbb{Q}(0) \to K_a \to \mathbb{Q}(-1) \xrightarrow{+1}$ $\in DMT(F)$!

§ The tannakian category $MT(F)$

Def. $DMT(F)^{\leq 0} \subset DMT(F)$ is the full subcategory consisting of iterated extensions of objects $\mathbb{Q}(-k)[-n]$ with $n \leq 0$

$DMT(F)^{\geq 0}$ — $n \geq 0$

Thm (Levine) $(DMT(F)^{\leq 0}, DMT(F)^{\geq 0})$ form a t-structure on $DMT(F)$.

Rem. Crucial point of proof: $\mathrm{Hom}(DMT(F)^{\leq 0}, DMT(F)^{\geq 0}[-1]) = 0$ which follows from

Cor. The heart $MT(F) := DMT(F)^{\leq 0} \cap DMT(F)^{\geq 0}$ is an ab. category, called the category of mixed Tate motives.

Rem: $MT(F)$ consists of iterated extensions of $\mathbb{Q}(-k)$, $k \in \mathbb{Z}$.

Consequence: $K_a \in MT(F)$, and we have a short exact seq. in $MT(F)$:

$$0 \to \mathbb{Q}(0) \to K_a \to \mathbb{Q}(-1) \to 0$$

From now on, $F = \mathbb{Q}$

Thm (Levine)

1) The category $MT(\mathbb{Q})$ is tannakian and contains the objects $\mathbb{Q}(-k)$ $\forall k \in \mathbb{Z}$. $\mathrm{Hom}_{MT(\mathbb{Q})}(\mathbb{Q}(-m), \mathbb{Q}(-n)) = \begin{cases} \mathbb{Q} \cdot id & \text{if } m=n \\ 0 & \text{else} \end{cases}$

2) Every $M \in MT(\mathbb{Q})$ has a canonical weight filtration W indexed by even integers, s.t.

$$\forall k, \quad gr^W_{2k} M := W_{2k} M / W_{2(k-1)} M \simeq \mathbb{Q}(-k)^{\oplus \alpha_k}$$

3) $\mathrm{Ext}^1_{MT(\mathbb{Q})}(\mathbb{Q}(-n), \mathbb{Q}(0)) \simeq (K_{2n-1}(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q})^{(n)} \simeq \begin{cases} \mathbb{Q}^\times \otimes_{\mathbb{Z}} \mathbb{Q} & n=1 \\ \mathbb{Q} & n \geq 3 \text{ odd} \\ 0 & \text{else} \end{cases}$

and $\mathrm{Ext}^n_{MT(\mathbb{Q})}(\mathbb{Q}(-n), \mathbb{Q}(0)) = 0 \quad \forall n \geq 2$

Rem: The category MHTS satisfies exactly the same structural properties, except for $\mathrm{Ext}^1_{MHTS}(\mathbb{Q}(-n), \mathbb{Q}(0)) \simeq \mathbb{C}/(2\pi i)^n \mathbb{Q}$, $\forall n \geq 1$

$$\begin{pmatrix} 1 & \lambda \\ 0 & (2\pi i)^n \end{pmatrix} \longleftrightarrow \lambda$$

We have de Rham / Betti realizations $\omega_{dR}, \omega_B : MT(\mathbb{Q}) \to \mathrm{Vect}_{\mathbb{Q}}$, and a Hodge realization $MT(\mathbb{Q}) \to MHTS$

It induces

$$\begin{array}{ccc} \mathrm{Ext}^1_{MT(\mathbb{Q})}(\mathbb{Q}(-n), \mathbb{Q}(0)) & \longrightarrow & \mathrm{Ext}^1_{MHTS}(\mathbb{Q}(-n), \mathbb{Q}(0)) \\ \wr\| & & \wr\| \\ K_{2n-1}(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q} & \xrightarrow[\text{regulator map}]{} & \mathbb{C}/(2\pi i)^n \mathbb{Q} \end{array}$$

Fact: This is injective

$n=1$: $\mathbb{Q}^\times \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \mathbb{C}/(2\pi i)\mathbb{Q}$ $\qquad \begin{pmatrix} 1 & \log(a) \\ 0 & 2\pi i \end{pmatrix}$

$a \otimes 1 \longmapsto \deg(a)$

$n \geq 3$ odd

$\mathbb{Q} \longrightarrow \mathbb{C}/(2\pi i)^n \mathbb{Q}$ $\qquad \begin{pmatrix} 1 & \zeta(n) \\ 0 & (2\pi i)^n \end{pmatrix}$ ($n \geq 3$ odd)

$1 \longmapsto \zeta(n)$

Consequence: 1) the regulator map is injective.

2) The Hodge realization $MT(\mathbb{Q}) \to MHTS$ is fully faithful (Goncharov)

Rem. We don't know how to construct these explicitly with a Betti representative.
We know they exist, and we can construct a big motive containing them, but we don't know how to take the convenient quotient of dim. 2.

$\forall v \in E_{2n+1, dR}$, $\langle \alpha, \mathrm{comp}(v) \rangle \in \mathbb{Q} + \mathbb{Q}\,\zeta(2n+1)$

If we had it explicit, we would have interesting linear forms in 1 and $\zeta(2n+1)$ $\rightsquigarrow$ prove (?) that $\zeta(2n+1) \notin \mathbb{Q}$.

For $n=1$, we know they (Brown) how to make it:

$$E_3 \simeq H^3(\overline{\mathcal{M}}_{0,6} - A, B - B\cap A) \rightsquigarrow \text{ this gives}$$

explicit integrals

$$\iiint_{[0,1]^3} \left(\frac{x(1-x)\,y(1-y)\,z(1-z)}{1-(1-xy)z} \right)^k \frac{dx\,dy\,dz}{1-(1-xy)z} \in \mathbb{Q} + \mathbb{Q}\,\zeta(3)$$

(Apéry - Bakers ~~motator~~ linear forms)

Rem. For F a general # field, we don't know how to compute the image of the regulator map in terms of known numbers. (related to ~~Zа~~ another Zagier's conjecture on polylogarithms)

Rem. $MT(\mathbb{Q})$ is too give, you wanna get rid of Kummer motives.

We define:

§ $\underline{MT(\mathbb{Z})}$

Def. (Deligne - Goncharov) $MT(\mathbb{Z}) \subset MT(\mathbb{Q})$ full subcategory consisting of obj. M s.t. every subquotient of M that is an extension of $\mathbb{Q}(-n)$ by $\mathbb{Q}(-n+1)$ is a split extension.

Fact. $MT(\mathbb{Z})$ has the same structural properties as $MT(\mathbb{Q})$, except for

$$\mathrm{Ext}^1_{MT(\mathbb{Z})}(\mathbb{Q}(-n), \mathbb{Q}(0)) \simeq \begin{cases} \mathbb{Q} & n \geq 3 \text{ odd} \\ 0 & \text{else} \end{cases}$$

You get rid of Kummer extensions.

Now, we want to understand the Galois groupoid $G_{dR} = \underline{\mathrm{Aut}}^{\otimes}(\omega_{dR})$

$\rightsquigarrow MT(\mathbb{Z}) \simeq \mathrm{Rep}(G_{dR})$

Prop. There is a semi-direct product decomposition $G_{dR} \simeq U_{dR} \rtimes \mathbb{G}_m$ where U_{dR} is pro-unipotent.

Pf. $p : G_{dR} \longrightarrow \mathrm{Aut}(\omega_{dR}(\mathbb{Q}(-1))) \simeq \mathbb{G}_m$

Define $U_{dR} = \ker(p)$,

$$1 \to U_{dR} \to G_{dR} \to \mathbb{G}_m \to 1$$

$\omega_{dR} : MT(\mathbb{Z}) \to \mathrm{gr}\,\mathrm{Vect}_{\mathbb{Q}} \rightsquigarrow \mathbb{G}_m \longrightarrow G_{dR}$ splitting for p.

For $M \in MT(\mathbb{Z})$, $U_{dR} \longrightarrow \mathrm{Aut}(\omega_{dR}(M)) \simeq GL_N$ the image $\mathcal{U}_{dR}(M)$ is unipotent wrt the weight filtration (because $\mathcal{U}_{dR}$ acts trivially on $\mathbb{Q}(-1)$)

Let's describe G_{dR} in three way:

- A Hopf algebra $\mathcal{H} = \mathcal{O}(\mathcal{U}_{dR})$ graded Hopf algebra

  $MT(\mathbb{Z}) \simeq \mathrm{grComod}(\mathcal{H})$

- Non-canonical isom. $\mathcal{H} \simeq (\mathbb{Q}\langle f_3, f_5, f_7, \dots \rangle, ш, \Delta)$

  non-comm. polynomials, f_{2n+1}, $\deg(f_{2n+1}) = 2n+1$

  (commutative) product

  coproduct given by $\Delta(f_3 f_5) = 1 \otimes f_3 f_5 + f_3 \otimes f_5 + f_3 f_5 \otimes 1$

  deconcatenation

- a Lie coalgebra $\mathcal{L} := \mathcal{H}_+ / \mathcal{H}_+ \mathcal{H}_+$ $\quad \mathcal{H}_+ = \bigoplus_{n>0} \mathcal{H}_n$

  indecomposables

  $\delta : \mathcal{L} \to \mathcal{L} \wedge \mathcal{L}$ cobracket. $\quad MT(\mathbb{Z}) \simeq \mathrm{gr}\,\mathrm{Comod}(\mathcal{L})$

  $\mathcal{L} \simeq \mathrm{Lie}^c(f_3, f_5, f_7, \dots)$ non-canonical

(we don't like coalgebras for psychological reasons, so dualize)

- a Lie algebra $\mathfrak{u}_{dR} := \mathcal{L}^\vee$

  $MT(\mathbb{Z}) \simeq \mathrm{gr}\,\mathrm{Rep}(\mathfrak{u}_{dR})$

  $\mathfrak{u}_{dR} \underset{\text{non-can.}}{\simeq} \mathrm{Lie}(\sigma_3, \sigma_5, \dots)$

<u>Exercise</u>. $\mathrm{Ext}^1_{\mathrm{gr}\,\mathrm{Rep}(\mathfrak{u}_{dR})}(\mathbb{Q}_n, \mathbb{Q}_0) \simeq (\mathcal{U}^{ab}_{-n})^\vee$ (abelianization)

$\mathbb{Q}_n$: trivial rep. of $\mathfrak{u}_{dR}$ in deg n.

$\mathrm{Ext}^1 \simeq K_{2n-1}(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}$

The only canonical thing is the image of σ_{2n+1} in $(\mathcal{U}_{dR})^{ab}$

§ Motivic periods for $MT(\mathbb{Z})$

$$\mathcal{P}^{mot} := \mathcal{O}\left(\mathrm{Isom}^{\otimes}(\omega_{dR}, \omega_B)\right) \xrightarrow{per} \mathbb{C}$$

↳ graded algebra

Def. Algebra of effective periods

$$\mathcal{P}^{mot,+} := \mathcal{O}\left(\mathrm{Isom}^{\otimes}_{\underbrace{MT^+(\mathbb{Z})}_{\subseteq MT(\mathbb{Z})}}(\omega_{dR}, \omega_B)\right)$$

$\subseteq MT(\mathbb{Z})$ consists of $M \in MT(\mathbb{Z})$ with non negative weights $W_2 M = 0$.

$$\mathcal{P}^{mot,+} \subset \mathcal{P}^{mot}$$

$$\mathcal{P}^{mot,+} = \bigoplus_{n \geq 0} \mathcal{P}^{mot,+}_n, \quad \mathcal{P}^{mot,+}_0 = \mathbb{Q}$$

$$\mathcal{P}^{mot} = \mathcal{P}^{mot,+}\left[\left((2\pi i)^{mot}\right)^{-1}\right]$$

Def. $\mathcal{P}^{mot,\mathbb{R}} :=$ the subsp. of $\mathcal{P}^{mot}$ given by fixed pts of complex conjugation

$$per : \mathcal{P}^{mot,\mathbb{R}} \longrightarrow \mathbb{R}$$

$$\mathcal{P}^{mot,+} = \mathcal{P}^{mot,+,\mathbb{R}} \oplus (2\pi i)^{mot} \mathcal{P}^{mot,+,\mathbb{R}}$$

Fact: ~~There is~~ $\cancel{\mathcal{P}}$ We have a coaction $\rho : \mathcal{P}^{mot} \to \mathcal{H} \otimes \mathcal{P}^{mot}$ "motivic Galois coaction". $\mathcal{P}^{mot,+,\mathbb{R}}$ is stable by ρ.

Fact. We have a projection

$$\begin{tikzcd} \mathcal{P}^{mot,+,\mathbb{R}} \arrow[r, "\rho"] \arrow[rrr, bend right, two heads, "\kappa"'] & \mathcal{H} \otimes \mathcal{P}^{mot,+,\mathbb{R}} \arrow[r] & \mathcal{H} \otimes \underbrace{\mathcal{P}^{mot,+,\mathbb{R}}_0}_{\mathbb{Q}} & \simeq \mathcal{H} \end{tikzcd}$$

$$\ker(K) = \left(\left((2\pi i)^{mot} \right)^2 \right)$$

<u>Prop</u>. We have a non-canonical isom.

$$\mathcal{P}^{mot,+,\mathbb{R}} \simeq \mathcal{H}\left[\left((2\pi i)^{mot} \right)^2 \right] \simeq \mathbb{Q}\langle f_3, f_5, f_7, \ldots \rangle \otimes \mathbb{Q}[\underset{\left((2\pi i)^{mot} \right)^2}{\underset{\parallel}{f_2}}]$$

which is compatible with the coaction

$$\mathcal{P}^{mot,+,\mathbb{R}} \longrightarrow \mathcal{H} \otimes \mathcal{P}^{mot,+,\mathbb{R}}$$

$$\rho(f_2) = 1 \otimes f_2$$

Pf. Use the fact that there exists an isom. $\omega_{dR} \simeq \omega_B$ defined over $\mathbb{Q}$.

$$\mathcal{P}^{mot} \simeq \mathcal{O}(G_{dR}) \simeq \mathcal{O}(\mathcal{U}_{dR}) \otimes \mathcal{O}(\mathbb{G}_m) \simeq \mathcal{H} \otimes \mathbb{Q}[t, t^{-1}]$$

$$(2\pi i)^{mot} \longleftrightarrow t$$

Add "+" and "R" $\square$

$$\begin{array}{lccc}
 & \mathcal{P}^{mot,+,\mathbb{R}} & \simeq & \mathbb{Q}\langle f_3, f_5, \ldots \rangle \otimes \mathbb{Q}[f_2] \\
 & \big\downarrow \text{ kill } ((2\pi i)^{mot})^2 & & \big\downarrow \text{ kill } f_2 \\
 & \mathcal{H} & \simeq & \mathbb{Q}\langle f_3, f_5, \ldots \rangle \\
 & \big\downarrow \text{ kill products} & & \big\downarrow \text{ kill products} \\
 & \mathcal{L} & \simeq & \mathrm{Lie}^c(f_3, f_5, \ldots) \\
 & \big\uparrow & & \big\uparrow \\
\ker \delta & \bigoplus_n K_{2n-1}(\mathbb{Z}) \otimes \mathbb{Q} & \simeq & \bigoplus_n \mathbb{Q} f_{2n+1}
\end{array}$$

Cor. $\dim\left(\mathcal{P}_n^{mot,+,\mathbb{R}}\right) = d_n$ ~ appearing in Zagier's conj

Pf. $$\sum_n \dim\left(\mathcal{P}_n^{mot,+,\mathbb{R}}\right) t^n = \frac{1}{1-(t^3+t^5+\dots)} \cdot \frac{1}{1-t^2} = \frac{1}{1-t^2-t^3} =$$

$$= \sum_n d_n t^n$$

§ Motivic MZVs

Remember for a word w in $\{0,1\}$, $I(0;w;1)$ was defined as an integral, $\zeta(n_1,\dots,n_r) = I(0;10\text{—}010\text{—}0 \text{———} 10\text{—}0;1)$

Extend this to $I(a_0; a_1 \dots a_n; a_{n+1})$ $\quad \forall a_i \in \{0,1\}$

s.t. $I(1;w;0) = (-1)^{\text{length } w} \, I(0; w^{\text{reversed}}; 1)$

and $I(a_0; \dots; a_{n+1}) = 0$ for $a_0 = a_{n+1}$, ≥ 1

and $I(a_0; a_1) = 1$

Thm. $\forall a_i$'s, there exists $I^{mot}(a_0; a_1 \dots a_n; a_{n+1}) \in \mathcal{P}_n^{mot,+,\mathbb{R}}$,

s.t. $\operatorname{per}(I^{mot}(a_0; a_1 \dots; a_{n+1})) = I(a_0; a_1 \dots a_n; a_{n+1})$

The construction satisfies the shuffle product formulas.

Thm. (Goncharov, Brown)

The infinitesimal coaction is given in these motivic periods by

$$D: \mathcal{P}^{mot,+,\mathbb{R}} \longrightarrow \mathcal{L} \otimes \mathcal{P}^{mot,+,\mathbb{R}}$$

$$D(I^{mot}(a_0; a_1 \dots a_n; a_{n+1})) = \sum_{0 \leq p < q \leq n} I^{\mathcal{L}}(a_p; \dots; a_{q+1}) \otimes I^{mot}(a_0; a_1, \dots a_p, a_{q+1}, \dots; a_{n+1})$$

Ex. $D(\zeta^{mot}(2,3)) = D(I^{mot}(0;10100;1)) =$

$= I^{\mathcal{L}}(0;10100;1)\otimes 1 + I^{\mathcal{L}}(0;101;0)\otimes I^{mot}(0;00;1) +$

$+ I^{\mathcal{L}}(1;010;0)\otimes I^{mot}(0;10;1) + I^{\mathcal{L}}(0;100;1)\otimes I^{mot}(0;10;1)$

$= \dots = \zeta^{\mathcal{L}}(2,3)\otimes 1 + 3\,\zeta^{\mathcal{L}}(3)\otimes\zeta^{mot}(2)$

Prop: $\delta: \mathbb{Q}\langle f_3, f_5, \dots\rangle[f_2] \longrightarrow \mathrm{Lie}^c(f_3, f_5, \dots)\otimes \mathbb{Q}\langle f_3,\dots\rangle[f_2]$

If X satisfies $D(X) = X^{\mathcal{L}}\otimes 1$, then $\begin{cases} X\in\mathbb{Q}\, f_{2n+1} & \text{if } \deg X = 2n+1 \\ X\in\mathbb{Q} f_2^n & \text{if } \deg X = 2n\end{cases}$

Prop $D\zeta^{mot}(n) = \zeta^{\mathcal{L}}(n)\otimes 1 \quad \forall n$

Cor. 1) We can choose an isom. $\mathcal{P}^{mot,+,\mathbb{R}} \simeq \mathbb{Q}\langle f_3,\dots\rangle[f_2]$

s.t. $\zeta^{mot}(2n+1)\longleftrightarrow f_{2n+1}$

2) $X\in\mathcal{P}^{mot,+,\mathbb{R}}$ hom. s.t. $D(X) = X^{\mathcal{L}}\otimes 1 \Rightarrow X\in\mathbb{Q}\,\zeta^{mot}(n)$

We have proved that $\zeta^{mot}(2n)\in\mathbb{Q}((2\pi i)^{mot})^{2n}$ (without any integral!)

$\zeta^{mot}(2n) = b_{2n}\,((2\pi i)^{mot})^{2n} \rightsquigarrow$

$\overset{per}{\rightsquigarrow} \zeta(2n) = b_{2n}\,(2\pi i)^{2n} \rightsquigarrow b_{2n} = (\dots)$ (Euler)

Thm (Brown) The motivic MZVs $\zeta^{mot}(n_1,\dots,n_r)$ with $n_i\in\{2,3\}$ are linearly independent.

Prop. $\zeta^{mot}(2,3)$ and $\zeta^{mot}(3,2)$ are not collinear:

Pf. $\bar{D}(X) = D(X) - X^{\mathcal{L}}\otimes 1$

$\bar{D}\,\zeta^{mot}(2,3) = 3\,\zeta^{\mathcal{L}}(3)\otimes\zeta^{mot}(2)$

$$\overline{D}\,\zeta^{mot}(3,2) = -2\,\zeta^{\mathcal{L}}(3) \otimes \zeta^{mot}(2)$$

Assume $\zeta^{mot}(2,3) = \lambda\, \zeta^{mot}(3,2)$, $\lambda \in \mathbb{Q}$

$\downarrow D$

$$3\,\zeta^{\mathcal{L}}(3) \otimes \zeta^{mot}(2) = -2\lambda\, \zeta^{\mathcal{L}}(3)\, \zeta^{mot}(2) \Rightarrow 3 = -2\lambda$$

$\overset{per}{\Rightarrow} \zeta(2,3) = -\frac{3}{2}\,\zeta(3,2)$ $\lightning$ (since MZVs are > 0)

$\square$

MOTIVES FOR PERIODS

28.08.2017

P. Jossen — Exponential motives and exponential periods

Intro

Numbers that are not periods:

$$e = \sum_{k \geq 0} (k!)^{-1}$$

It doesn't seem that it has a representation as an integral.

But we don't have any concrete example of a complex numbers which are not periods, even if almost all numbers aren't.

We will show here that $(e)^{mot}$ is not a motivic period.

(I) Introduce cohomology for (X, f), X variety, function $f: X \to \mathbb{A}^1$
de Rham and Betti.

(II) Motives for pairs (X, f) following Nori ⇝ exponential motives

(III) More realisation

(IV) Period conjecture and consequences

§1. Rapid decay (co)homology

Let $X/\mathbb{C}$ be any ~~variety~~ variety, $Y \subseteq X$ closed subvariety.

$f: X \to \mathbb{A}^1$ $f \in \mathcal{O}_X(X)$ - potential

$$\underbrace{H^{rd}_n}_{\text{rapid decay}}(X, Y, f) = \lim_{t \to \infty} H^{sing}_n(X(\mathbb{C}), Y(\mathbb{C}) \cup f^{-1}(S_t))$$

$$S_t = \{ z \in \mathbb{C} \mid Re(z) \geq t \}$$

Similarly,

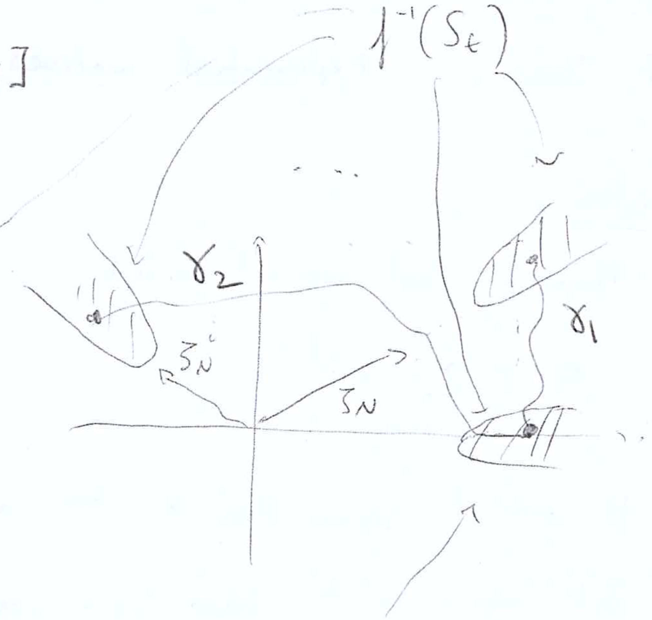
$$H^n_{rd} \, \mathbb{Q} = \operatorname{colim}_{t \to \infty} H^n_{sing}(\ldots)$$

Coefficients always $\mathbb{Q}$.

These are f.d. vect. sp., and the limit $\lim_{t \to \infty}$ stabilizes, i.e. $t \gg 0$ gives constant (co)homology

Ex: $X = \mathbb{A}^1$, $f = f(x) \in \mathbb{C}[x]$
$$\overset{\shortparallel}{x^N + \cdots}$$

$H^1_{rd}(X, f)$



Then $H_1^{rd}(X,f)$ is generated by paths between these regions.

$\gamma_1, \dots, \gamma_{N-1} \in H_1^{rd}(X,f)$ are a basis.

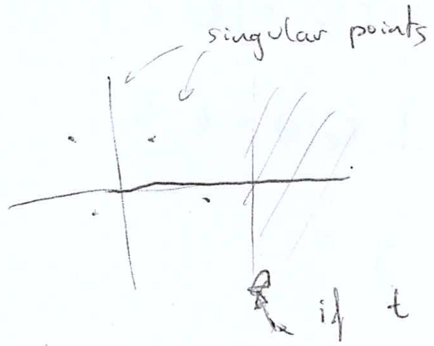
Fix X, Y, f

$$X \times \mathbb{A}^1 \supseteq (Y \times \mathbb{A}^1) \cup \Gamma_f \qquad p^{-1}(t) = (X, Y \cup f^{-1}(t))$$
$$\downarrow$$
$$\mathbb{A}^1$$

Set $\beta : (X \times \mathbb{A}^1) \setminus ((Y \times \mathbb{A}^1) \cup \Gamma_f) \hookrightarrow X \times \mathbb{A}^1$ inclusion

$R^n p_*(\beta_! \beta^* \underline{\mathbb{Q}}_{X \times \mathbb{A}^1})$ = a constructible sheaf on $\mathbb{A}^1$.

$F(S_t) = H^n(X(\mathbb{C}), Y(\mathbb{C}) \cup f^{-1}(S_t))$



if t is in this part, $F(S_t)$ stabilizes.

Functoriality

$$\begin{array}{ccc} \varphi: X & \longrightarrow & X' \\ \cup\!\!\!\mid & & \cup\!\!\!\mid \\ \varphi|_Y\ Y & \dashrightarrow & Y' \end{array} \qquad \begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & \mathbb{A}^1 & \end{tikzcd}$$

We get $\varphi^*: H^n_{rd}(X', Y', f') \longrightarrow H^n_{rd}(X, Y, f)$

Cup product:

$$H^n(X, Y, f) \otimes H^{n'}(X', Y', f') \to H^{n+n'}(X \times X', Y \times X' \cup X \times Y'; f \boxplus f')$$

where $(f \boxplus f')(x, x') = f(x) + f'(x')$

Künneth formula.

§ De Rham cohomology for (X, f)

Let k char $= 0$, X/k variety smooth, ~~affine~~ $f: X \to \mathbb{A}^1_k$ potential (i.e. a regular function)

not usual differential, twisted by f.

$$H^n_{dR}(X, f) = \mathbb{H}^n_{Zar}(X, (\Omega^\bullet_X, d_f))$$

$$d_f(\omega) = d\omega - df \wedge \omega, \qquad d_f \circ d_f = 0$$

Notice if f is constant, $d_f = d$.

If X is affine, $H^n(X, \Omega^p_X) = 0$, $\quad n \geq 1$

$\Rightarrow H^n_{dR}(X, f)$ is computed by

$$\begin{array}{l} \Omega^0(X) \xrightarrow{d_f} \Omega^1(X) \longrightarrow \cdots \\ \shortparallel \\ \mathcal{O}_X(X) \end{array}$$

There is a unique way of extending sensibly the definition to $H^n_{dR}(X, Y, f)$ for arbitrary X, Y.

<u>Example</u>: $X = \mathbb{A}^1$, $Y = \phi$, $f = f(x) = x^N + \dots \in k[x]$
$N \geq 1$

$$k[x] \xrightarrow{d_f} k[x]\,dx$$

$d_f(g) = g'\,dx - (f'g)\cdot dx$ is injective

A basis for H^1_{dR} is given by the forms $dx, x\,dx, \dots, x^{N-2}dx$

$H^1(X, f) \simeq k^{N-1}$

§ <u>Comparison isomorphism</u>, $k \subseteq \mathbb{C}$

We produce a special isom.

$$\alpha : H^n_{dR}(X, Y, f) \otimes_k \mathbb{C} \longrightarrow H^n_{rd}(X, Y, f) \otimes_{\mathbb{Q}} \mathbb{C}$$

Regard α as a pairing

Integration, k-linear, $\mathbb{Q}$-linear

$$I : H^n_{dR}(X, Y, f) \otimes H^{rd}_n(X, Y, f) \longrightarrow \mathbb{C}$$

Define I for X smooth, affine, $Y = \phi$

Fix X, f, n. $H^n_{dR}(X, f) \otimes H^{rd}_n(X, f) \to \mathbb{C}$

$[\omega] \in H^n_{dR}(X, f)$, $\omega \in \Omega^n(X)$, $d_f\omega = 0$

$(\gamma_t)_{t \gg 0} \in H^{rd}_n(X, f)$, $\gamma_t \in H^{sing}_n(X(\mathbb{C}), f^{-1}(S_t))$

n-cycle $\Delta^n \to X(\mathbb{C})$ s.t. $\partial\gamma_t \subseteq f^{-1}(S_t)$

$$I([\omega], [\gamma_t]) = \lim_{t \to \infty} \int_{\gamma_t} \omega \cdot e^{-f} \in \mathbb{C}$$

Here $\lim_{t \to \infty} \int_{\gamma_t} \omega$ tends to diverge, since γ_t becomes larger and larger. But only where $\mathrm{Re}(f) \gg 0$, so we add e^{-f} to control this. decays rapidly in the direction of γ_t getting bigger. The name "~~rapid~~ decay" comes from this

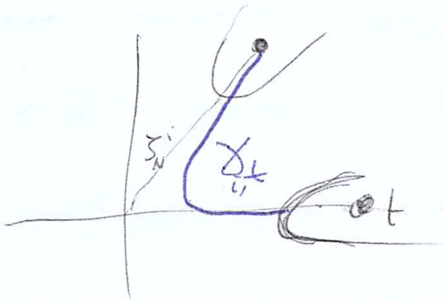
Thm (Sabbah, Hien-Roucairol)

$I([\omega],[(\gamma_t)])$ is well defined and induces the sought isom, $\alpha : H^n_{dR}(X,Y,f) \otimes \mathbb{C} \longrightarrow H^n_{rd}(X,Y,f) \otimes_{\mathbb{Q}} \mathbb{C}$

This isom. is functorial and compatible with cup products

Ex: $H^1_{dR}(\mathbb{A}^1, f) = k\langle dx, x\, dx, -, x^{N-2} dx \rangle$, $f \in k[x]$, $f = x^N + ..$

$H_1^{rd}(\mathbb{A}^1, f) = \mathbb{Q}\langle \gamma_1, -, \gamma_{N-1} \rangle$



Then $I([x^{j-1}dx],[\gamma_i]) =$

$$= \lim_{t\to\infty} \int_{\gamma_{i,t}} x^{j-1} e^{-f(x)} dx = -\int_0^\infty x^{j-1} e^{-f(x)} dx$$

$$+ \int_0^\infty (\zeta^i x)^{j-1} e^{-f(\zeta^i x)} d\zeta^i x$$

Choose $f(x) = x^N$

$$I([x^{j-1}dx], (\gamma_i)) = \frac{\zeta^{ij}-1}{N} \cdot \Gamma\left(\frac{j}{N}\right)$$

This will be the period matrix.

(older) Conjecture (Lang, Rohrlich) $N \geq 3$. Then,

$$\operatorname{tr deg}_{\mathbb{Q}} \overline{\mathbb{Q}}\left(\Gamma\left(\tfrac{1}{N}\right), \Gamma\left(\tfrac{2}{N}\right), -, \Gamma\left(\tfrac{N-1}{N}\right)\right) = \frac{\varphi(N)}{2} + 1$$

Euler function

Known: $\leq$, $=$ if $\varphi(N) = 2$, $N = 3,4,6$

Rem. For $N=2$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Hence the motive should be something that ~~multiplied~~ tensored by itself, gives you the motive corresponding to π (i.e. $\mathbb{Q}(1)$), so is something like $\mathbb{Q}(\frac{1}{2})$

This motive is $H^1(\mathbb{A}^1, x^2)$.

A Hodge structure will have to satisfy that its square is the Hodge structure of $\mathbb{Q}(1)$. Later more on this.

Examples: $X = \operatorname{Spec} k$, $Y = \emptyset$, $f = c$ cst

$H^0_{dR}(X, f) = k \ni 1$

$H_0^{r.d}(X, f) = \mathbb{Q} \ni 1$ $\qquad \int_1 1 \cdot e^{-c} = e^{-c}$

Exponentials of elements of k are exp periods.

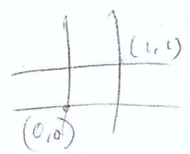
Theorem Lindemann-Weierstrass: $c_1, \dots, c_n \in \bar{\mathbb{Q}} \subset \mathbb{C}$.

Then $\operatorname{trdeg}_{\bar{\mathbb{Q}}}(\bar{\mathbb{Q}}(e^{-c_1}, \dots, e^{-c_n})) = \dim_{\mathbb{Q}}(\operatorname{span}_{\mathbb{Q}}(c_1, \dots, c_n))$

i.e., linear relations on the right give algebraic relations on left.

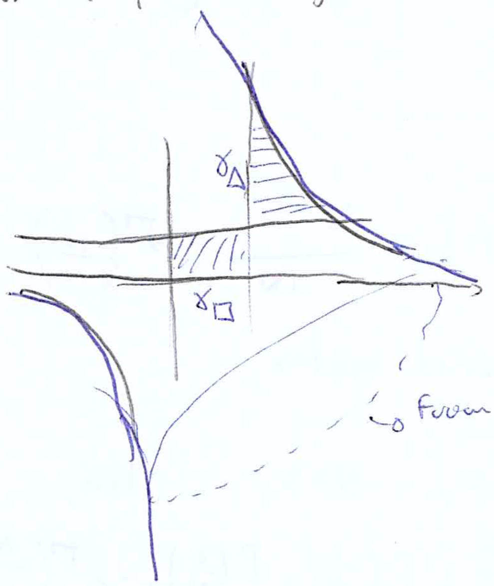
Example $X = \mathbb{A}^2 = \operatorname{Spec} \mathbb{Q}[x, y]$

$Y = \{x = 0 \text{ or } x = 1 \text{ or } y = 0 \text{ or } y = 1\}$ $\rightsquigarrow$



$f(x, y) = x \cdot y$

$H_2^{r.d}(X, Y, f)$



$f(x,y) = t$ real, large

$\hookrightarrow$ From complex picture, γ_0

$\gamma_0 : [0,1]^2 \to X(\mathbb{C}) = \mathbb{C}^2$

$(r, \varphi) \mapsto (r t e^{2\pi i \varphi}, r \cdot e^{-2\pi i \varphi})$

Then $H_2^{r.d}(X, Y, f) = \mathbb{Q}\langle \gamma_\square, \gamma_\Delta, \gamma_0 \rangle$

$H^2_{dR}(X, Y, f) = k^3 \ni dx\, dy$

$$I(\omega, \gamma_\square - \gamma_\Delta) = \left(\int_\square - \int_\Delta\right) e^{-xy}\,dx\,dy = \int_0^1\int_0^1 e^{-xy}\,dx\,dy = \gamma$$

Brosnan Belkale

Euler-Mascheroni constant

So this constant is an exponential period (it seems Kontsevich conjectured it wouldn't be)

$\zeta(s) = \frac{1}{s-1} + \gamma + \text{h.o.t}$

Open: γ irrational?

Today: we construct motives for varieties with potential

§1. Quivers and representations

Def. A *quiver* Q consists of a class of objects $Ob(Q)$ and a class of morphisms $Mor(Q)$ and functions

$$\begin{tikzcd} Ob(Q) \arrow[r, "i"] & Mor(Q) \arrow[l, bend right, "s"'] \arrow[l, bend left, "t"] \end{tikzcd} \qquad \left.\begin{array}{l} s = \text{source} \\ t = \text{target} \\ i = \text{identity} \end{array}\right\} \quad s\circ i = id = t \circ i$$

Rem Can regard Q as a category without composition law. In particular categories are quivers. Adopt terminology "functor", "natural transform", "full sub-quiver".

Def A quiver is *finite* if $Ob(Q)$, $Mor(Q)$ are finite sets

Def Let F be a field. A repr. of Q is a functor

$\rho : Q \longrightarrow Vect_F \rightsquigarrow$ fin. dim F-v.s.

A morphism of quiver repr.: $(Q,\rho) \to (Q',\rho')$ is a function $\varphi : Q \to Q'$ together with a η transform

$$\begin{tikzcd} Q \arrow[rr, "\varphi"] \arrow[dr, "\rho"'] & & Q' \arrow[dl, "\rho'"] \\ & \mathrm{Vect}_F & \end{tikzcd}$$

$\rho' \circ \varphi \simeq \rho$ isomorphism

Given $Q \xrightarrow{\rho} \mathrm{Vect}_F \rightsquigarrow$ we construct an F-linear abelian category $\cancel{\langle Q, \rho \rangle}$ $\mathrm{End}(Q, \rho)$

as follows: obj. = f.d. F-v.s. together with an action $\mathrm{End}(\rho) \xrightarrow{\alpha} \mathrm{End}(V)$

$\mathrm{End}(\rho)$: F-alg.

$$\prod_{q \in \mathrm{Ob}(Q)} \mathrm{End}_F(\rho(q))$$

s.t. $\exists\ Q_0 \subseteq Q$ finite subquiver and factorisation

$$\begin{tikzcd} \mathrm{End}(\rho) \arrow[r, "\text{restr.}"] \arrow[rr, bend left, "\alpha"] & \mathrm{End}(\rho|_{Q_0}) \arrow[r, "\alpha_0"] & \mathrm{End}(V) \end{tikzcd}$$

f. dim. F-alg.

i.e. α continuous w.r.t. pro-structure

Morphisms = F-lin. morph. comp. with actions

Composition = comp. of F-lin. maps.

$$\mathrm{End}(\rho) = \lim_{\substack{Q_0 \subseteq Q \\ \text{fin.}}} \mathrm{End}(\rho|_{Q_0}) \longrightarrow \text{pro-f.d. } F\text{-algebra}$$

$$\langle Q, \rho \rangle = \begin{pmatrix} F. \text{ dim. } & F. \text{ v.sp} \\ + \text{ cont. } & \mathrm{End}(\rho) \text{ action} \end{pmatrix}$$

$$\begin{tikzcd} & \langle Q, \rho \rangle \arrow[d, "\text{forget}"] \\ Q \arrow[ur, dashed, "\tilde{\rho}"] \arrow[r, "\rho"'] & \mathrm{Vect}_F \end{tikzcd}$$

$\tilde{\rho}(q) =$ v.sp. $\rho(q)$ + obvious action $\mathrm{End}(\rho) \to \mathrm{End}(\rho(q))$

Thm (Nori)

Consider

$$\begin{tikzcd}
 & \mathcal{A} \arrow[dr, bend left, "f"] & \\
 & \langle Q,\rho\rangle \arrow[u, dashed, "\Phi\ \exists!"] \arrow[dr] & \\
Q \arrow[uur, bend left, "\sigma"] \arrow[ur, "\tilde{\rho}"] \arrow[rr, "\rho"'] & & \mathrm{Vect}_F
\end{tikzcd}$$

$\mathcal{A}$ abelian, F-linear

f exact, faithful, F-linear

Φ unique up to unique isom. of functors.

Φ exact, F-linear

- Main ingredients of proof.

1) Establish functoriality of $(Q,\rho) \longrightarrow \langle Q,\rho\rangle$ for morph. of quiver repr.

2) Look at σ as morphism $(Q,\rho) \to (\mathcal{A},f)$, considered as a quiver

$$\begin{tikzcd}
\langle Q,\rho\rangle \arrow[r, "\Sigma"] \arrow[dr, dashed, "\Phi"] & \langle \mathcal{A},f\rangle \\
Q \arrow[u, "\tilde{\rho}"] \arrow[r, "\sigma"'] & \mathcal{A} \arrow[u, "\tilde{f}"']
\end{tikzcd}$$

3) show $\tilde{f}$ is an equiv. of cat., similar to Mitchell-Freyd's thm.

---

§ Exponential motives $\quad k \subseteq \mathbb{C}$ subfield

$Q^{exp}(k)$ is the quiver $\rho : Q^{exp}(k) \longrightarrow \mathrm{Vect}_{\mathbb{Q}}$ the repr. defined as follows:

Obj: Q^{exp} are tuples $[X, Y, f, n, i]$

X k-variety, $Y \subseteq X$ closed, $f: X \to \mathbb{A}^1$

$\rho([X,Y,f,n,i]) = H^n_{rd}(X,Y,f)(i)$,

where the first $-(i)$ means

$\mathbb{Q}(1) = H_2(\mathbb{P}^1) = H_1(\mathbb{G}_m)$

$V(i) = V \otimes \mathbb{Q}(1)^{\otimes i}$

Morphisms in Q^{exp} with target $[X, Y, f, n, i]$

a) for
$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & \mathbb{A}^1 & \end{tikzcd}$$
, $\varphi(Y) \subseteq Y'$

$$\varphi^* : [X', Y', f', n, i] \to [X, Y, f, n, i]$$

$P(\varphi^*) = \varphi^* =$ morphism induced in r.d. cohom.

b) For every closed $Z \subseteq Y$, a morphism

$$\partial : [Y, Z, f|_Y, n-1, i] \to [X, Y, f, n, i]$$

$P(\partial) =$ connecting morphism in the long ex. seq. $Z \subseteq Y \subseteq X$.

c) A morphism

$$\kappa : [X \times \mathbb{G}_m, Y \times \mathbb{G}_m \cup X \times \{1\}, f \boxplus 0, n+1, i+1]$$
$$\downarrow$$
$$[X, Y, f, n, i]$$

$P(\kappa) =$ Künneth morphism (iso)

$\mathcal{M}^{exp}(k) = \langle Q^{exp}(k), P \rangle$ ab. $\mathbb{Q}$-linear

Forgetful functor: $\mathcal{M}^{exp} \xrightarrow{R_B} Vect_{\mathbb{Q}}$

Betti realisation (faithful, exact, conservative

Illustration of universal property:

$\mathcal{P}$ = cat. of triples (V, W, α)

$V =$ k-v.sp.
$W = \mathbb{Q}$-v.sp
$\alpha : V \otimes_k \mathbb{C} \xrightarrow{\sim} W \otimes_{\mathbb{Q}} \mathbb{C}$

Ab. $\mathbb{Q}$-linear (tannakian). $\mathcal{P}$ cat. of_n period structures.

$$
\begin{tikzcd}
 & + \arrow[dr, bend left, "{(V,W,\alpha)\mapsto W}"] & \\
 & \mathcal{M}^{exp}(k) \arrow[u, dashed] \arrow[dr, "R_B"'] & \\
Q^{exp}(k) \arrow[uur, bend left, "\sigma"] \arrow[ur, "\tilde{\rho}"] \arrow[rr, "\rho", "\text{r.d. coh.}"'] & & \mathrm{Vect}_{\mathbb{Q}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{P} \\
\mathcal{M}^{exp}(k) \arrow[u, dashed, "{R_{per} = \text{period realization}}"']
\end{tikzcd}
$$

$$\sigma(X, Y, f, n, i) = (V, W, \alpha)$$

V: $H^n_{dR}(-)$, W: $H^n_{r.d}(-)$, α comp. isom.

§ The idea of cellular pairs

$X = U \cup V$ covering by 2 opens $\quad f : X \to \mathbb{A}^1$

Mayer - Vietoris

$$\underset{\circlearrowleft\, \mathrm{End}(\rho)}{H^{n-1}(U \cap V, f|_{U\cap V})} \xrightarrow{\partial} \underset{\circlearrowleft\, \mathrm{End}(\rho)}{H^n(X, f)}$$

$$[U \cap V, \phi, f|_{\dots}, n-1, 0] \dashrightarrow [X, \phi, f, n, 0]$$

__Def__. Call $[X, Y, f, n, i]$ __cellular__ if

- $H^p(X, Y, f) = 0$ for $p \neq n$
- X affine of dim $\leq n$

Example $[\mathbb{G}_m, \{1\}, 0, 1, i]$.

$$[\mathbb{A}^1, Y, \underset{\text{constant}}{\underset{\neq}{f}}, 1, i]$$

__Thm__ (Beilinson, Nori) Let X be affine of dim $\leq n$, $Y \subseteq X$ closed, f. There exists $X_0 \subseteq X_1 \subseteq \dots \subseteq X_n = X$ closed, dim $X_p \leq p$ s.t. each $Y_p = X_p \cap Y$, $[X_p, Y_p \cup X_{p-1}, f, p, i]$ is cellular. Moreover, any given filtration respecting dim. condition can be refined to such a cellular filtration.

Let X be affine, $\dim \leq n$, $Y \subseteq X$, $f : X \to \mathbb{A}^1$

Choose a cellular filtration $X_\bullet$ as theorem

$$\cdots \to H^p(X_p, Y_p \cup X_{p-1}, f|_{X_p}) \xrightarrow{\partial} H^{p+1}(X_{p+1}, Y_{p+1} \cup X_p, f|_{X_{p+1}})$$

$\|$ $\quad \ni \mathcal{M}^{exp}$ $\quad$ morphism in $\mathcal{M}^{exp}$ $\quad \ni \mathcal{M}^{exp}$

def

$C^\bullet(X, Y, f)$ $\rightsquigarrow$ Here $H_q(C^\bullet(X,Y,f)) = H^q(X, Y, f)$ because there is a spectral sequence, and choosing stuff that is cellular ~~degenerates~~ implies that it degenerates

This is an obj. in $D^b(\mathcal{M}^{exp})$, indep. of choices $(X_\bullet)$

Given $U \cup V = X$ can now ~~die~~ consider

$$0 \to C^\bullet(X, f) \to C^\bullet(U, f|_U) \oplus C^\bullet(V, f|_V) \to C^\bullet(U \cap V, f|_\cap) \to 0$$

We want a tensor product in $\mathcal{M}^{exp}(k)$ (even ~~the~~ tannakian cat.)

§ Tensor product on $\mathcal{M}^{exp}$

Thm $\exists !$ $\otimes$-structure on $\mathcal{M}^{exp}$ s.t.

(1) $R_B : \mathcal{M}^{exp} \longrightarrow \mathrm{Vect}_{\mathbb{Q}}$ is strictly compatible with $\otimes$

(2) Cup products

$$H^n(X, Y, f)(i) \otimes H^{n'}(X', Y', f')(i') \to H^{n+n'}(X \times X', \ldots, f \boxplus f')(i+i')$$

are morph. of motives.

With respect to this $\otimes$-structure, $\mathcal{M}^{exp}$ is tannakian and R_B is a fiber functor.

How to construct $\otimes$? On cell. objects, there is just one choice.

Fact. Let $Q^{exp}_c \subseteq Q^{exp}$ full subquiver of cellular objects. Then inclusion induces an equiv. of categories of linear envelopes

$$\langle Q^{exp}_c, \rho|_{Q^{exp}_c} \rangle \to \langle Q^{exp}, \rho \rangle = \mathcal{M}^{exp}$$

Let (Q, ρ), (Q', ρ') be quiver repr.

$$Q \boxtimes Q' = \text{quiver with } \begin{cases} \text{obj} - (q, q') \text{ pairs} \\ \text{morph.: } (f, f') : (q_1, q_1') \to (q_2, q_2') \\ \quad \text{and } f = id \text{ or } g = id \end{cases}$$

$$\rho \boxtimes \rho'(q, q') = \rho(q) \otimes \rho'(q')$$

Prop. $\mathrm{End}(\rho \boxtimes \rho') \simeq \mathrm{End}(\rho) \otimes \mathrm{End}(\rho')$ isom. of $\mathbb{Q}$-alg.

Apply to

$$\begin{tikzcd} Q^{exp}_c \boxtimes Q^{exp}_c \arrow[rr, "T"] \arrow[dr, "\rho \boxtimes \rho'"'] & & Q^{exp}_c \arrow[dl] \\ & \mathrm{Vect}_{\mathbb{Q}} & \end{tikzcd}$$

On morphisms, T can be defined because of the condition

Künneth: $\rho \circ T \overset{K}{=} \rho \boxtimes \rho$

Get: $\mathrm{End}(\rho) \longrightarrow \mathrm{End}(\rho \boxtimes \rho) = \mathrm{End}(\rho) \otimes \mathrm{End}(\rho)$ comultiplication.

This gives $\otimes$. Reformulate thm.

$(\mathrm{End}(\rho), \text{comult})$ is a Hopf-alg. $\mathcal{M}^{exp}$ = f.d. vect. sp +

+ $\mathrm{End}(\rho)$ action = f.d. comodules of $A = \mathrm{Hom}(\mathrm{End}(\rho), \mathbb{Q})$ =
commutative Hopf alg.

$=$ f.d. repr. of $G^{mot,exp} = \mathrm{Spec}(A)$

$M \in \mathcal{M}^{exp}$ correspo. to a repr. $G^{mot} \to GL_V$,

$V =$ f.d. $\mathbb{Q}$-v.sp. $= R_B(M)$

$G^{mot}_M =$ image of this repr.

<u>Period conjecture:</u> $\mathrm{tr\,deg}$ (Periods of M) $= \dim G^{mot}_M$

<u>Hard problem:</u> In $\mathcal{M}^{exp}(k)$ we have $\mathbb{Q}(n) = H^0(\mathrm{Spec}\, k, \phi, o)(n)$

$= H^{2n}(\mathbb{P}^n, \phi, o)$ one dimens. period $= (2\pi i)^n$

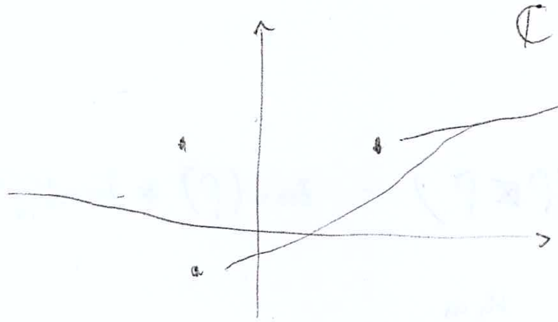
Compute $\mathrm{Ext}^1_{\mathcal{M}^{exp}}(\mathbb{Q}(-n), \mathbb{Q})$

For $n=1$, it should be $(k^\times \otimes \mathbb{Q}) \oplus \mathbb{Q}$

<u>§ the category Perv_0</u>

Perv_0 is the categ. of constructible sheaves F of $\mathbb{Q}$-v.sp. on $\mathbb{A}^1$ s.t. $H^*(\mathbb{A}^1, F) = 0$

How to give such an F?



$\mathbb{C}$

- $S =$ singularities
- Local system on $\mathbb{C} \setminus S =$
  $=$ a vsp V + autom. $\gamma_s : V \to V$ which are the local monodromies.
- Vector sp. $(V_s)_{s \in S}$ + maps $\alpha_s : V_s \to V^{\gamma_s}$

$$0 \longrightarrow \bigoplus_{s\in S} V_s \oplus V \xrightarrow{\ d\ } \bigoplus_{s\in S} V \longrightarrow 0$$

$$((v_s), v) \longmapsto (\alpha_s(v_s) - v)_{s\in S}$$

This complex computes $H^*(\mathbb{A}^1, F)$. $H^*(--)=0 \iff$

$\iff$ 1) α_s are inj.

2) $\bigcap_{s\in S} \alpha_s(V_s) = \{0\}$

3) $\sum_{s\in S} \underbrace{V/\alpha_s(V_s)}_{\text{vanishing cycles @ } s} = \dim(V)$

<u>Ex</u> : $S = \{s\}$.

Local system $\underline{V}$ constant on $\mathbb{C}\setminus\{s\}$, $V_s = 0$

Given $F, G \in \mathrm{Perv}_0$, define additive convolution

$F * G = R^! \mathrm{sum}_*(p_1^* F \otimes p_2^* G) \overset{!}{\in} \mathrm{Perv}_0$

$$\mathbb{A}^2 \xrightarrow{p_1, p_2} \mathbb{A}^1$$

<u>Prop</u>. $(\mathrm{Perv}_0, *)$ is tannakian $\mathbb{Q}$-linear

$F^\vee = \tau^* \mathbb{D} F$, $\quad \tau(x) = -x$

A fiber functor is $\Psi_\infty(F) = \lim_{t\to\infty} F(S_t)$

§ Perverse realisation

$$\begin{tikzcd}
 & \mathrm{Perv}_0 \arrow[ddr, "\Psi_\infty"] & \\
 & \mathcal{M}^{exp} \arrow[u, dashed, "R_{Perv}"] \arrow[dr, "R_B"] & \\
Q^{exp} \arrow[uur, bend left, "\sigma"] \arrow[ur] \arrow[rr, "\rho \text{ r.d. cohom,}"'] & & \mathrm{Vect}_{\mathbb{Q}}
\end{tikzcd}$$

Given $[X, Y, f, n, i]$

$$X \times \mathbb{A}^1 \supseteq Y \times \mathbb{A}^1 \cup \Gamma_f$$
$$\downarrow pr$$
$$\mathbb{A}^1$$

$\beta : (X \times \mathbb{A}^1) \setminus (Y \times \mathbb{A}^1 \cup \Gamma_f) \hookrightarrow X \times \mathbb{A}^1$

$$\sigma([X, Y, f, n, i]) = R^n pr_* (\beta_! \beta^* \mathbb{Q}_{X \times \mathbb{A}^1})(i) \in \mathrm{Perv}_0$$

This gives R_{Perv}

We had

$$G^{exp} = \mathrm{Aut}^{\otimes}(R_B : \mathcal{M}^{exp} \to \mathrm{Vect}_{\mathbb{Q}})$$

$$G^{per} = \mathrm{Aut}^{\otimes}(\omega_B : \mathcal{P} \to \mathrm{Vect}_{\mathbb{Q}})$$
$$(V, W, \alpha) \longmapsto W$$

$$G^{Perv} = \mathrm{Aut}^{\otimes}(\Psi_\infty)$$

$$\begin{tikzcd}
 & \mathcal{M}^{exp} \arrow[dl, "R_{Perv}"'] \arrow[dr, "R_{Per}"] & \simeq \mathrm{Rep}(G^{exp}) \\
\mathrm{Perv}_0 & & \mathcal{P} \simeq \mathrm{Rep}(G^{per})
\end{tikzcd}$$

$$G^{Perv} \nearrow G^{exp} \nwarrow G^{per}$$

For a single object
$M \in \mathcal{M}^{exp}(k)$
this translates into $\quad G_M^{exp} := \mathrm{im}(G^{exp} \to GL_{R_B(M)}) \subseteq GL_d$, $\quad d = \dim M$

$G_M^{per} := \mathrm{im}(G^{per} \to GL_{R_B(M)})$

$G_M^{Perv} := \mathrm{im}(G^{Perv} \to GL_{R_B(M)})$ $\quad / \quad G_M^{perv} \subseteq G_M^{exp} \supseteq G_M^{per}$

Let $k/\mathbb{Q}$ algebraic

Conj (Formal period conj.). The period realisation

$$\mathcal{M}^{exp} \longrightarrow \mathcal{P}$$

is full. (We know exact and faithful) ~~In particular,~~ Equivalently

$$G_M^{per} = G_M^{exp} \quad \forall M.$$

Rem. Open, but checked in many examples, thanks to group theory.

Fix $M \in \mathcal{M}^{exp}$, consider: $\mathbb{P} = R_{per}(M)$

$\langle P \rangle^{\otimes} \subseteq \mathcal{P}$ tann. cat. generated by $\mathbb{P}$.

$G_M^{per} = \mathrm{Aut}^{\otimes}(\omega_B : \langle P \rangle^{\otimes} \longrightarrow \mathrm{Vect}_{\mathbb{Q}})$

$T_M = \mathrm{Isom}^{\otimes}(\omega_B \otimes k, \omega_{dR})$ period torsor. This is a

$(G_M^{per})_k$ - torsor.

$T_M(\mathbb{C}) \ni \alpha_M =$ the comp. isom.

Call $\mathcal{O}_{T_M}$ the ring of formal periods of M.

Actual periods are $k[\text{coeff of } A \text{ and } \det A^{-1}]$, for the matrix A of α wrt some bases $\rightsquigarrow A_M$,

Prop. The evaluation map $\mathcal{O}_{T_M} \longrightarrow A_M : f \longmapsto f(\alpha)$

is surj. and $\mathrm{Spec}\, A_M \subseteq T_M$ is the Zariski closure$/k$ of α.

Conj. (Trascendence) $\forall M \in \mathcal{M}^{exp}$, the following ~~hold~~ equiv. statements hold:

1) The ev. map $\mathcal{O}_{T_M} \longrightarrow A_M$ is inj. (an isom.)

2) $\alpha \in T_M(\mathbb{C})$ is dense

3) $\mathrm{Spec}\, A_M \subseteq T_M$ is a G_M^{per} - torsor

Status: open, very few evidence

Rem. The classical period conjecture is the conjunction of these two conjectures.

Ex. Exponentials of alg. numbers

Pick $c \in k$, define a motive (exponential)

$$E(c) = H^0(\operatorname{Spec} k, \emptyset, -c)(0) \in \mathcal{M}^{exp}(k), \qquad \dim E(c) = 1$$

$R_B(E(c)) = \mathbb{Q}$

$R_{dR}(E(c)) = k$

$$\alpha : k \otimes \mathbb{C} \longrightarrow \mathbb{Q} \otimes \mathbb{C}$$
$$1 \otimes 1 \longmapsto 1 \otimes e^c$$

$\Downarrow$

Period Matrix is (e^c).

$$A_{E(c)} = k[e^{\pm c}]$$

$$G^{exp}_{E(c)} \subseteq GL_{R_B(E(c))} \simeq GL_1$$

The tannakian category gen by $E(c)$ is

$$E(c) \otimes E(c) = H^0(\operatorname{Spec} k, \emptyset, -c) \otimes H^0(\operatorname{Spec} k, \emptyset, -c) =$$

because no higher cohom. and therefore cellular

$$= H^0(\operatorname{Spec} k, \emptyset, -2c)$$

$$E(c) \otimes E(c') = E(c+c')$$

1^{st} possibility: $N \geq 1$, $E(c)^{\otimes N} \simeq E(0)$ $\quad (\Rightarrow G^{exp}_{M(c)} \subseteq \mu_N)$

2^{nd} possibility: $E(c)^{\otimes N} \simeq E(0) \Leftrightarrow N = 0 \Rightarrow G^{exp}_{M(c)} = GL_1$

Now we use the perverse realisations $R_{Perv}(E(c)) = \mathcal{E}(c) \in \mathrm{Perv}_0$

$\mathbb{A}^1 \times \operatorname{Spec} k \supseteq \{-c\}$, $\quad \mathcal{E}(c) = R^0 p_*(\beta_! \beta^* \mathbb{Q}_{\mathbb{A}^1 \times \operatorname{Spec} k}) =$

$\downarrow p$

$\mathbb{A}^1$

$=$ constant sheaf $\mathbb{Q}$ on $\mathbb{A}^1 \setminus \{-c\}$ extended by zero to $\mathbb{A}^1$

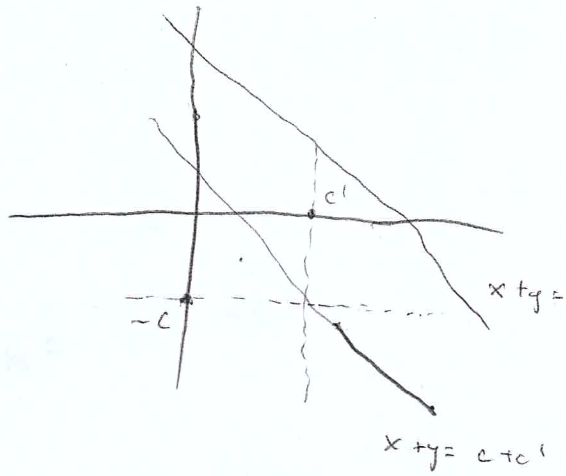
In Perv_0, $\mathcal{E}(c) * \mathcal{E}(c') = \mathcal{E}(c+c')$

$$\parallel$$

$$R^1 sum_* \left(pr_1^* \mathcal{E}(c) \otimes pr_2^* \mathcal{E}(c') \right)$$

$$\mathbb{A}^2 \xrightarrow[sum]{pr_1\ pr_2} \mathbb{A}^1$$

Here c is the singularity of the sheaf



$\mathcal{E}(c) = \mathcal{E}(c') \Leftrightarrow c = c'$

$\mathcal{E}(c)^{\otimes N} \simeq \mathcal{E}(0) \Leftrightarrow N \cdot c = 0$

$$\Rightarrow \quad \begin{matrix} G_M^{Perv} \\ \cap \\ G_M^{exp} \end{matrix} = \begin{cases} \mathbb{G}L_1 & \text{if } c \neq 0 \\ \{1\} & \text{if } c = 0 \end{cases}$$

- Period conjecture: e^c is trascendental for $c \neq 0$

More general, $M = E(c_1) \oplus E(c_2) \oplus \cdots E(c_n)$

$R_{Perv}(M) = \mathcal{E}(c_1) \oplus \cdots \oplus \mathcal{E}(c_n)$

$\langle R_{perv}(M) \rangle^{\otimes} = \langle$ sums of $\mathcal{E}(c) \mid c \in \mathbb{Z}$-span of $c_1, \ldots, c_n \rangle$

$G_M^{perv} =$ the torus dual to $\mathbb{Z}$-span of $c_1, \ldots, c_n$ $= G_M^{exp}$

Period conj: $\quad$ tr deg $(\mathbb{Q}(e^{c_1}, \ldots, e^{c_n})) = \text{rank } \mathbb{Z} \text{ span } (c_1, \ldots, c_n)$

$$\uparrow$$

Weierstrass Thm

<u>Example</u>: Assume period conj. Then e is not a classical period.
Indeed, e would be an elt of the period algebra of a classical

~~periods~~ motive $M_0 = H^n(X, Y, f=0)(i)$

set $M = M_0 \oplus E(1)$.

Enough to show: $\dim M > \dim M_0$.

$R_{perv}(M_0)$:

$$\begin{array}{c} \mathbb{A}^1 \times X \supseteq \mathbb{A}^1 \times Y \cup \{0\} \times X \\ \downarrow p \\ \mathbb{A}^1 \end{array}$$

$$R^n p_* \left(\beta_! \beta^* \underline{\mathbb{Q}}_{\mathbb{A}^1 \times X}\right)(i) = \text{the constant sheaf } H^n(X,Y)(i)$$

outside $\{0\}$, extended by zero to $\mathbb{A}^1$.

$\simeq \mathcal{E}(0) \otimes H^n$

$G^{perv}_{M_0} = \{1\}$

but $G^{perv}_M \longrightarrow G^{perv}_{E(1)} = GL_1 \Rightarrow G^{perv}_M = G^{perv}_{E(1)}$

in the kernel

$$\begin{array}{ccccc} GL_1 \hookrightarrow G^{exp}_M & \longrightarrow & G^{exp}_{M_0} & & \\ \cup\mid & & \cup\mid & & \\ GL_1 \cong G^{perv}_M & \longrightarrow & G_{M_0} = \{1\} & \longrightarrow & 0 \end{array}$$

diagram commutative

$\Rightarrow \dim G^{exp}_M > \dim G^{exp}_{M_0}$

$\Downarrow$ conjecture

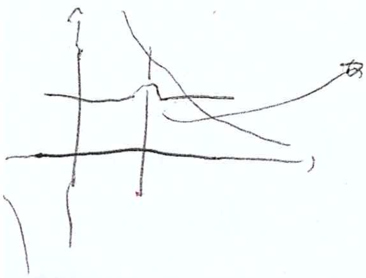
$$\operatorname{tr\,deg}(A_M) > \operatorname{tr\,deg}(A_{M_0})$$

$$\| \quad A_M = A_{M_0}[e, e^{-1}]$$

$\Rightarrow$ e transcendental $/ A_{M_0}$

<u>Example</u>: Euler–Mascheroni γ. It appears in

$H^2(X, Y, f)$, $X = \mathbb{A}^2$, $Y = xy(x-1)(y-1) = 0$, $f(x,y) = xy$



blow-up at $(1,1)$, so we merge the cycles $\gamma_\square$ and γ_Δ.

$$R_B(M) = H^2_{rd}(\tilde{X}, \tilde{Y}, f) = \mathbb{Q}(\gamma_\square - \gamma_\Delta) \oplus \gamma_0$$

$$R_{dR}(M) = k\, dx\, dy \oplus k\, \delta \qquad \delta = \delta_\infty \in \Omega^0(\{0,0\})$$

Period matrix:

	δ	$dx\,dy$
$\gamma_\square - \gamma_\Delta$	1	γ
γ_0	0	$2\pi i$

$$A_M = k[2\pi i^{\pm 1}, \gamma]$$

Here we don't know if γ is irrational, so looking at Betti realization doesn't help. (In the previous example we could have done that). So we look at R^{perv}_M

$$X \times \mathbb{A}^1 \supseteq Y \times \mathbb{A}^1 \cup \Gamma_f$$

$$\begin{array}{c} \downarrow p \\ \mathbb{A}^1 \end{array}$$

$$R^2 p_*(-)_t = H^2(X, Y \cup f^{-1}(t)), \quad \underbrace{\mathbb{Q}(\gamma_\square - \gamma_\Delta)^{(t)}}_{\substack{\text{here trouble}\\ \text{when } t=0,\\ \text{so we get monodromy}}} \oplus \underbrace{\mathbb{Q}(\gamma_0)^{(t)}}_{\substack{\text{this part}\\ \text{is constant wrt}\\ t.}}$$

There the hyperbola degenerates in 2 lines.

One singular point $t=0$

Monodromy u : $u(\gamma_\square - \gamma_\Delta)^{(1)} = (\gamma_\square - \gamma_\Delta)^{(1)} + \gamma_0^{(1)}$

$u(\gamma_0^{(1)}) = \gamma_0^{(1)} \quad \rightsquigarrow \quad u = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

__Fact__. Let $\mathcal{L}_1, \mathcal{L}_2$ be local systems on $\mathbb{A}^1 \setminus \{0\}$,

$j : \mathbb{A}^1 \setminus \{0\} \hookrightarrow \mathbb{A}^1$

$$j_! \mathcal{L}_1, \; j_! \mathcal{L}_2 \in \mathrm{Perv}_0, \quad j_! \mathcal{L}_1 \overset{\text{convolution}}{*} j_! \mathcal{L}_2 = j_!(\mathcal{L}_1 \otimes \mathcal{L}_2)$$

$\Rightarrow u \in G^{perv}_M$, so G^{perv}_M contains the matrices $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$

Moreover, they are all.

$$0 \to \mathbb{Q} \to M \to \mathbb{Q}(-1) \to 0$$

$$\rightsquigarrow$$

$$G_M^{mot} = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}$$

from perverse realisation

from classical realisations

$\neq 0$

Period conj $\Rightarrow$ $2\pi i$ and δ are alg. indep.

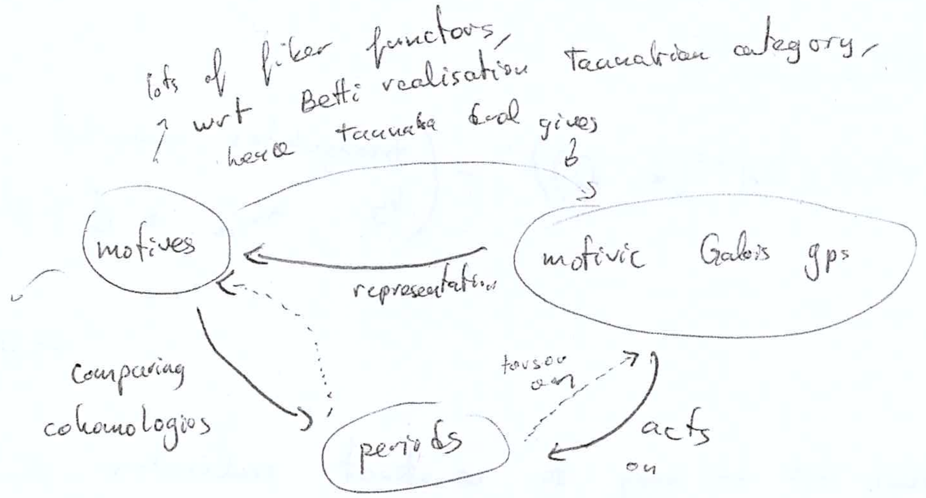
01.09

M. Gallauer Alves de Souza

Motivic Galois gps for motives for periods

① Introduction

$k \subseteq \mathbb{C}$



lol

This action was used by C. Dupont to give relations between periods.

The picture is very conjectural

② Nori motives

Quiver $Q(k)$, vertices $\to$ (X, Y, n, i) (homology)

edges $\to$ - $(X, Y, n, i) \to (X', Y', n, i)$ for each $(X, Y) \to (X', Y')$

- connecting morph.: $(X, Y, n, i) \to (Y, \emptyset, n-1, i)$
- Tate twist

Nori motives:

$M_N(k) := \langle Q(k), \underbrace{\omega_B}_{\text{Betti realisation}} \rangle$

$\omega_{dR}: M_N(k) \to \mathrm{Mod}(k)$

$$\begin{tikzcd}
M_N(k) \arrow[rr, leftrightarrow, "\text{Tannaka duality}"] \arrow[dr] & & G_N(k) \arrow[dl] \\
& P_N(k) := \mathcal{O}(\mathrm{Isom}^{\otimes}(\omega_B, \omega_{dR})) &
\end{tikzcd}$$

Conjecture (Nori, Kontsevich): $\mathcal{P}_{KZ}(k) \simeq \mathcal{P}_N(k)$

Now proved.

Rem. $M_N(k)$ is too difficult to work with.

(3) Voevodsky motives

$$DM(k) := (DA^{\acute{e}t}(k;\mathbb{Q})) = \left(\begin{array}{l}\text{triangulated monoidal category comp. gen.}\\ \text{by } Sm_{/k} + \mathbb{Q}(-1)\end{array}\right)$$

(ét-descent, $\mathbb{A}^1$ contractible)

Rem. Here not so easy to construct realisation functors.

$$Sm_{/k} \ni X \longmapsto \mathrm{Sing}(X^{an})$$

$$Re_B : DM(k) \longrightarrow D(\mathbb{Q})$$

$$Re_{dR} : DM(k) \longrightarrow D(k)$$

Voevodsky motives
$$M_V(k) = DM(k)^{\omega} \quad \text{(constructible motives)}$$

~~DM(k)~~

derived gp scheme $/\mathbb{Q}$

$$\begin{tikzcd}
DM(k) \arrow[dr] & & G_V^{der}(k) \arrow[ll, bend right, "\text{derived Tannakian dual}"'] \arrow[ll, dashed, "???"] \arrow[dl] \\
& \mathcal{P}_V^{der}(k) &
\end{tikzcd}$$

- We don't know how to see $DM(k)$ as the derived version of a Tannakian category.
- The arrow $\overset{?}{\dashleftarrow}$ is related ~~with~~ to the conservativity conjecture
- We want to construct an actual gp scheme (not just the derived ~~fu~~ version) and we do this with a "π_0". $G_V(k)$

Similarly for $\mathcal{P}_V^{der}(k) \longrightarrow \mathcal{P}_V(k)$

- We should then have

$$\begin{tikzcd} P_V(k) \arrow[r, dashed, bend left] & G_V(k) \arrow[l, bend left] \end{tikzcd}$$

- <u>Fact</u>. 1) There are canonical (iso-) morphisms $G_V^{\Delta}(k) \simeq G_V^{qm}(k) \simeq$
$\simeq G_V^{dg}(k)$ of pro-alg gps $/\mathbb{Q}$

2) $$G_V^{\infty}(k) \longrightarrow G_V^{\Delta}(k)$$ is not ~~necessarily~~ a priori an isom.

$$\Pi_0\left(aut^{\otimes}(Re_B^{\infty})\right) \longrightarrow aut^{\otimes}\,\Pi_0\left(Re_B^{\infty}\right) = aut^{\otimes} Re_B^{\Delta}$$

3) Assume $\exists$ motivic t-structure.

Then $$G_V^{Lev}(k) \text{ "="} G_V(k)$$

$G_V^{\bullet}(k)$ all the same

$$\underbrace{M_V(k)^{\heartsuit}}_{\text{heart}} = Rep\left(G_V(k)\right)$$

$$M_V(k) = D^b\left(Rep\left(G_V(k)\right)\right)$$

(4) Voevodsky vs Nori motives

$Sm\,Aff_{/k} \ni X \rightsquigarrow$ cellular filtration

$$X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_0$$

$$H_n(X_n, X_{n-1}) \to H_{n-1}(X_{n-1}, X_{n-2}) \to \cdots$$

$$\in Cpx\left(M_N(k)\right)$$

Functor

$$C : DM(k) \longrightarrow D\left(Ind\; M_N(k)\right)$$

$$C^{w} : M_V(k) \longrightarrow D^b\left(M_N(k)\right)$$

<u>Thm</u> (Choudhury - G.) $\mathcal{C}$ induces an isomorphism of pro-alg. gps $/\mathbb{Q}$

$$G_V(k) \xrightarrow{\cong} G_N(k)$$

<u>Rem</u>. If $\mathcal{C}^\omega$ is an equiv., this would be an easy consequence.

<u>Pf</u>. (Sketch)

Universal property of Ayoub's mot. Gal. gp:

0) $DM(k) \xrightarrow{\mathcal{C}} D(\mathrm{Ind}\ M_N(k)) \to$ forget comod. structure

$\to \mathrm{Comod}\ D(\mathbb{Q})\ (\mathcal{O}(G_N(k))) \to D(\mathbb{Q})$

and we get $\boxed{\mathcal{O}(G_V^{der}(k)) \xrightarrow{c} \mathcal{O}(G_N(k))}$

$$H_0(\mathcal{O}(G_V^{der}(k))) \nearrow$$
$$\shortparallel$$
$$\mathcal{O}(G_V(k))$$

1) $Q(k) \xrightarrow{R} \cancel{DM(k)}\ M_V(k)$

$(X, Y, n) \longmapsto \mathrm{cone}(M(Y) \to M(X))[n]$

$Q(k) \to M_V(k) \to \mathrm{CoMod}(\mathcal{O}(G_V^{der}(k))) \xrightarrow[H_0]{} \mathrm{CoMod}(\mathcal{O}(G_V(k))) \to \mathrm{Mod}(\mathbb{Q})$

ωB

$M_N(k) \to \mathrm{CoMod}(\mathcal{O}(G_V(k)))$

$\Leftrightarrow \boxed{\mathcal{O}(G_N(k)) \xrightarrow{r} \mathcal{O}(G_V^{der}(k))}$

2) $cr = id$

3) $rc = id$

$\square$

<u>Cor</u> $P_V(k) \simeq P_N(k) \simeq P_{KZ}(k)$

we have a nice description of this in terms of holomorphic functions given by Ayoub, so we get a nice descript. of