# ALGEBRAIZATION THEOREMS: FROM GAGA TO FOLIATIONS OVER NUMBER FIELDS

TATIHOU ISLAND, 4-8 JULY 2016

**Monday.**

9:00 Welcome + Introduction 1-2

9:45 Talk 1
Algebraization in complex analytic geometry I. From Puiseux to Chow. 3-7
Yohan Brunebarbe

10:45 Coffee break

11:15 Talk 2
Algebraization in complex analytic geometry I. Géométrie Algébrique et Géométrie Analytique. 7-13
Michele Ancona

12:15 Lunch break

14:15 Talk 3 13-19
The formalism of slopes. Vector bundles on projective curves.
Diego Izquierdo

15:15 Coffee break

15:45 Talk 4
The Hodge Index Theorem on projective surfaces. Connectivity theorems. 19-23
Frank Gounelas

16:45 Break (no coffee)

17:15 Talk 5 23-28
An introduction to formal geometry.
Nicola Mazzari

**Tuesday.**

9:30 Talk 6
The theorems of Andreotti and Hartshorne. 29-33
Florent Martin

10:30 Coffee break

11:00 Talk 7
The theorems of Grauert and Grothendieck (SGA2).
Olivier Benoist 33-39

12:15 Lunch break

14:30 Talk 8
Algebraic foliations. 39-43
Arne Smeets

15:45 Coffee break

16:15 Talk 9 43-49
Some geometric applications.
Simon Pepin Lehalleur

**Wednesday.**

9:30 Talk 10
First steps in Arakelov geometry. 49-55
Ramla Abdellatif

10:30 Coffee break

11:00 Talk 11
Arakelov geometry on arithmetic surfaces. 55-64
Ariyan Javanpeykar

12:15 Lunch break
Free afternoon (there will be coffee at some point)

19:00 Additional talk
Theta series, infinite rank Euclidean lattices, and Diophantine algebraization
Jean-Benoît Bost

**Thursday.**

9:30 Talk 12
Schwarz Lemma. 65-69
Robert Kucharczyk

10:30 Coffee break

11:00 Talk 13
Arithmetic algebraization à la Chudnovsky I. Statements and proofs. 69-74
Peter Jossen

12:15 Lunch break

14:30 Talk 14
Arithmetic algebraization à la Chudnovsky II. Diophantine applications. 75-80
Ziyang Gao

15:45 Coffee break

16:15 Talk 15
Arithmetic algebraization à la Schneider-Lang I. Statements and proofs. 80-82
Lars Kühne

**Friday.**

9:00 Talk 16
Arithmetic algebraization à la Schneider-Lang II. Diophantine applications. 82-85
François Charles

10:15 Coffee break

10:30 Additional talk
Cycles in the de Rham cohomology of abelian varieties over number fields. 85-89
Yunqing Tang

04.07.2016

# I. GAGA

## 1. From Puiseux to Chow.

### 0. Intro. Algebraization theorems

E.g. : • Hermite, Lindemann $\quad \mathbb{Q} \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$

$$e \notin \overline{\mathbb{Q}}, \quad \pi \notin \overline{\mathbb{Q}}$$

$$\{\alpha \in \mathbb{C} \mid \alpha, \exp(\alpha) \in \overline{\mathbb{Q}}\} = \{0\}$$

• Severi

$$\begin{tikzcd} \mathbb{P}^1_{\mathbb{C}} \arrow[rr, hook] & & \mathbb{P}^N_{\mathbb{C}} \\ & \mathcal{U} \arrow[ul, hook] \arrow[ur, dashed, "\varphi"] & \end{tikzcd}$$

$\varphi \in \mathcal{M}(\mathcal{U})$ meromorphic.

Then $\mathcal{M}(\mathcal{U}) = \mathbb{C}(x_1, \dots, x_N)$

Aim of workshop: understand that both statements are essentially the same.

**Basic ideas**

(1) "Auxiliary polynomials"

If we are interested in $A \subset \mathbb{C}^N \hookrightarrow \mathbb{P}^N(\mathbb{C})$, consider $\overline{A}^{\text{Zariski closure}}$

$I :=$ ideal def. Zariski closure.

For $d \in \mathbb{N}$, $\quad I_d = I \cap \mathbb{C}[x_1, \dots, x_N]_{\leq d} \simeq \Gamma(\mathbb{P}^N_{\mathbb{C}}, \mathcal{I}\,\mathcal{O}(d))$

$$\mathbb{C}[x_1, \dots, x_N]_{\leq d} / I \simeq \Gamma(\overline{A}^{Zar}, \mathcal{O}(d)) \quad \dim \sim d^{\dim \overline{A}^{Zar}}$$

Example: $A$ = compact Riemann surf. in $\mathbb{P}^2(\mathbb{C})$

$$\Gamma(\overline{A}^{Zar}, \mathcal{O}(d)) \hookrightarrow \Gamma^{an}(A, \mathcal{O}(d)) \quad \mathrm{rk}_{\mathbb{C}} - \lesssim d$$

$\dim \overline{A}^{Zar} \leq 1$

(2) Analogy between function fields and # fields.

Dedekind, Weber $\rightsquigarrow$ algebraic approach to Riemann's results

$\mathbb{Q} \hookrightarrow \mathbb{Z}$ $\qquad\qquad$ $k(T) \hookleftarrow k[T]$

$[K:\mathbb{Q}] < \infty$, $\mathcal{O}_K$ fin. extension $\qquad$ $\underset{K}{\overset{\shortparallel}{k(C)}} \hookleftarrow k[C \setminus \{P\}]$

$\hookrightarrow$ sm proj. curve

$$\begin{array}{ccc} X & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \text{model} \\ \operatorname{Spec} K & \longrightarrow & \operatorname{Spec} \mathcal{O}_K \end{array}$$

$$\begin{array}{ccc} X & \longrightarrow & \mathfrak{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \longrightarrow & C \end{array}$$

sections here $\qquad$ sections here

give

[ - Something similar with sections in LHS, but ⚠ $\mathcal{O}_K$ not proper.
One looks to the "hermitian geom. of $X_{\mathbb{C}}$." ]

$\overline{E} = (E, \|\cdot\|)$ euclidean lattice $\qquad$ $E$ vect. bundles in $\mathcal{C}$

deg $E \in \mathbb{Z}$

Rem. $\qquad$ algebraic $\longmapsto$ analytical

analytical $\longmapsto$ algebraic

Formal germs on $X$ around $P \in \mathfrak{X}(K)$

give us results here $\leftarrow$ Statements à la Serre

Eg. $\qquad\qquad$ $\mathbb{P}^1 \xrightarrow{\text{restrict}} \mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$

$\mathbb{Q} \qquad \mathbb{Z}$ $\qquad\qquad$ $\mathcal{O}(d) \rightsquigarrow \mathcal{O}_{\mathbb{A}^1}$

generating section $s \in \Gamma(\mathcal{O}_{\mathbb{A}^1})$

$E = \mathbb{Z}$ $\qquad$ $s$ rational section of $\mathcal{O}_{\mathbb{P}^1}$

Generating section is 1.

Looking at $(s) \rightsquigarrow \operatorname{val}_\infty(s) = d$

Reference: Huayi Chen (Paris),
lecture notes in Arakelov Geom., chapters 3, 4, 5. $\rightsquigarrow$ Ask Marla if needed

1 – Puiseux to Chow

Aim Explain

Thm (Chow, '49) Any closed $\mathbb{C}$-analytic subvariety in $\mathbb{P}^N(\mathbb{C})$ is algebraic

(1) Rank of evaluation maps and dim of Zariski closures

$F \subset \mathbb{C}^N$

$\forall k \geq 1, \quad \eta_k : \mathbb{C}[x_1, \ldots, x_N]_{\leq k} \to \mathcal{C}(F, \mathbb{C})$

Set $\overline{F}^{Zar} \subset \mathbb{P}^N(\mathbb{C})$

Prop. $\mathrm{rk}(\eta_k) \underset{k \to \infty}{\sim} \dfrac{\deg(\overline{F})}{\dim(\overline{F})!} \cdot k^{\dim(\overline{F})}$

Pf. $\mathbb{C}[x_1, \ldots, x_N]_{\leq k} \xrightarrow{\eta_k} \mathcal{C}(F, \mathbb{C})$

$P(1, x_1, \ldots, x_N)$

$P(x_0, \ldots) \quad \mathbb{C}[x_0, \ldots, x_N]_k \longrightarrow \mathbb{C}[x_0, \ldots, x_N] / I(F)_k$

$\mathrm{rk}(\eta_k) = \dim\left(\mathbb{C}[x_0, \ldots, x_N] / I(F)_k\right) \rightsquigarrow$ given by Hilbert polyn.

□

(2) An algebraic criterion for analytic submanifolds

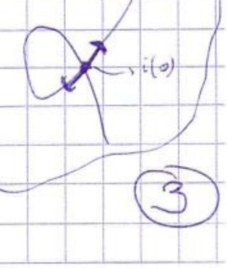
Let $i : \mathbb{B}^d \longrightarrow \mathbb{P}^N(\mathbb{C})$ an analytic embedding
$\mathbb{B}^d \subset \mathbb{C}^d$

and $F = i(\mathbb{B}^d) \subset \mathbb{P}^N(\mathbb{C})$

Lem. $\overline{F}^{Zar}$ is an irr. alg. variety s.t. $\dim \overline{F}^{Zar} \geq d$.

Pf. irr. $\rightsquigarrow$ $F$ conn. complex manifold + analytic continuation

dim $\rightsquigarrow$ $F \not\subset \overline{F}_{sing} \Rightarrow \underbrace{F \cap \overline{F}_{reg} \neq \emptyset}_{\text{non-trivial}}$ (e.g. $\overline{F}^{Zar}$, $i(0)$ the node




3

Prop (2) Assume $\exists\, c>0$ s.t. $\forall k\in\mathbb{N}$, $\forall s\in\Gamma(\mathbb{P}^N(\mathbb{C}),\mathcal{O}(k))$,

$$i^*s\neq 0 \Rightarrow \mathrm{mult}_0(i^*s)\leq c\cdot k.$$

Then $\dim(\mathcal{F})=d$

Recall. Let $\mathcal{O}_0^{an}$ = germs of analytic functions on $\mathbb{B}^d$.

$\cup$

$\mathfrak{m}$ max. ideal.

$\forall f\in\mathcal{O}_0^{an}$, $\mathrm{mult}_0(f)=\sup\{k \mid f\in\mathfrak{m}^k\}\in\mathbb{N}\cup\{\infty\}$

Pf. Estimate $\mathrm{rk}(\eta_k)$

$$E_k := \Gamma(\mathbb{P}^N(\mathbb{C}),\mathcal{O}(k)) \simeq \mathbb{C}[x_0,\dots,x_N]_k$$

Define a non-increasing filtration:

$$s\in E_k^l \Leftrightarrow \mathrm{mult}_0(i^*s)\geq l$$

Then, let $(z_1,\dots,z_d)$ local coord's of $\mathbb{B}^d$. Fix a trivialization of $i^*\mathcal{O}(k)$.

We get a map

$$E_k^l \longrightarrow \mathbb{C}[z_1,\dots,z_d] : s\mapsto \text{degree } l \text{ term of Taylor exp. around } 0$$

factors $E_k^l / E_k^{l+1}$

Now $\mathrm{rk}(\eta_k)=\dim_{\mathbb{C}}\left(E_k \Big/ \bigcap_{l\geq 0} E_k^l\right) = \sum_{l\geq 0}\dim_{\mathbb{C}}(E_k^l/E_k^{l+1})$

Since $E_k^{\lfloor c\cdot k\rfloor}=E_k^{\lfloor c\cdot k\rfloor+1}=\cdots=0$, this is a finite sum ~~equal to~~ $\leq$

$$\leq \sum_{l=0}^{\lfloor c\cdot k\rfloor}\dim_{\mathbb{C}}\left(\mathbb{C}[z_1,\dots,z_d]_l\right)=\dim\left(\mathbb{C}[z_1,\dots,z_d]_{\leq\lfloor ck\rfloor}\right)=$$

$$=\binom{\lfloor c\cdot k\rfloor+d}{d}$$

Hence, $\dim(\bar{F}) \le d$

By previous lemma, we're done.

$\square$

Proof of Chow's thm in the smooth case

Prop ③ $X$ conn. closed complex submanif. of $\mathbb{P}^N(\mathbb{C})$.

Fix $P \in X$.

$\forall k \ge 0, \quad \forall s \in \Gamma(\mathbb{P}^N(\mathbb{C}), \mathcal{O}(k)), \quad s|_X \ne 0 \Rightarrow \mathrm{mult}_P(s|_X) \le \deg(X) \cdot k$

where $\deg(X)$ is the topological degree.

Recall: topological degree: $X \subset \mathbb{P}^N(\mathbb{C})$

$$H^{2i}(\mathbb{P}^N(\mathbb{C}), \mathbb{Z}) = \mathbb{Z} \cdot [H]^i$$

$$[X] \in H^{2 \cdot \mathrm{codim}(X)}(\mathbb{P}^N(\mathbb{C}), \mathbb{Z})$$
$$= \deg(X) \cdot [H]^{\mathrm{codim}\, X}$$

$X \subset \mathbb{P}^N(\mathbb{C})$ connected submanifold of $\dim = d$

$P \in X, \quad i : \mathbb{B}^d \longrightarrow X \subseteq \mathbb{P}^N(\mathbb{C})$ an embedding

$\overline{i(\mathbb{B}^d)}^{Zar} = \bar{X}^{Zar}$ is an irred. alg. variety.

Prop $2+3 \Rightarrow \dim(\bar{X}) = d = \dim(X)$

Claim. $\bar{X}_{reg} \cap X$ is a non-empty open and closed subset of $\bar{X}_{reg}$ (in the analytic top.).

Hence, $\bar{X}_{reg} \cap X = \bar{X}_{reg} \Rightarrow \bar{X}_{reg} \subseteq X \Rightarrow \bar{X} \subseteq X$

~~Pf of claim.~~ Let $X$ be an irr. complex variety. We are using

(i) $X_{reg}(\mathbb{C})$ is dense in $X(\mathbb{C})$ for analytic top.

(ii) $X(\mathbb{C})$ is connected

Pf of claim · Non-empty: empty $\Leftrightarrow X \subseteq \bar{X}_{sing} \subsetneq \bar{X}$ ↯

· Closed because $X$ closed

· Open: $\bar{X}_{reg}$, $X$ are complex submanifolds of $\mathbb{P}^N(\mathbb{C})$,

~~one contai~~ $\underbrace{\bar{X}_{reg} \cap X \subseteq \bar{X}_{reg}}_{\text{same dimension}} \Rightarrow$

$\Rightarrow \bar{X}_{reg} \cap X$ open complex submanifold of $\bar{X}_{reg}$

□

Still to prove: Prop 3.

Pf. $X$ connected, complex submanifold of $\mathbb{P}^N(\mathbb{C})$, $P \in X$

$\forall k \geq 0$, $s \in \Gamma(\mathbb{P}^N(\mathbb{C}), \mathcal{O}(k))$

$s|_X \neq 0 \Rightarrow \text{mult}_P\, s|_X \leq \deg(X) \cdot k$.

Step 1: $\dim = 1 \rightsquigarrow$ some stuff with Riemann surfaces

$$\text{mult}_P(s|_X) \underset{\text{by def}}{\leq} \deg(s|_X) \qquad s|_X \in \Gamma(X, \mathcal{O}(k)|_X)$$

$$= \deg_X(\mathcal{O}(k)|_X)$$

$$= k \deg(\mathcal{O}(1)|_X)$$ and top. degree = alg. degree

Step 2: arbitrary dim $\rightsquigarrow$

$Gr = \{L \subset \mathbb{P}^N(\mathbb{C}),\ (N-d+1)\text{-plane containing } P\}$

$\tau = \{L \mid \forall x \in X \cap L,\ T_x L + T_x X = T_x \mathbb{P}^N(\mathbb{C})\} \subset Gr$

Lemma ① $\tau \subset Gr$ open dense subset

② $\{T_P(L \cap X),\ L \in \tau\}$ is an open dense subset of $\mathbb{P}(T_P X)$

③ $\forall k \geq 0, \quad \forall s \in \Gamma(\mathbb{P}^N(\mathbb{C}), \mathcal{O}(k))$

$$\text{mult}_p(s|_X) = \min_{L \in \mathcal{T}} \text{mult}_p(s|_{X \cap L})$$

Taylor expansion gives $\leq$, ② gives $<$ happens in a closed subset $\underset{+②}{\Rightarrow}$ $=$.

□

c.f. "Topology differential point of view," Milnor for singularity issues,

GAGA (Serre '56)

$X$ proj. variety $/\mathbb{C}$, $\mathcal{O}_X$ = regular functions

$X^h$ analytification, $\mathcal{H}_X$ = holom. functions

$$\left\{ \begin{array}{c} \text{algebraic coh. sheaves} \\ \text{on } X \end{array} \right\} \xrightarrow{\text{eq.}} \left\{ \begin{array}{c} \cancel{\text{alg}} \text{ analyt. coh.} \\ \text{sheaves on } X^h \end{array} \right\}$$

preserving Čech cohom. gps.

Construction: ① $\mathcal{F}$ alg. coh. sheaf, $id: X^h \to X$ is cont., $\mathcal{F}' = id^{-1}\mathcal{F}$.

$$\mathcal{F}' \otimes_{\mathcal{O}'} \mathcal{H} =: \mathcal{F}^h$$

Fact: $\mathcal{F}$ alg. coh. sheaf $\Rightarrow$ $\mathcal{F}^h$ analytic coherent sheaf.

② $\varphi: \mathcal{F} \to \mathcal{G}$ alg. morph. $\rightsquigarrow$ $\varphi^h: \mathcal{F}^h \to \mathcal{G}^h$ analytic morphism.

Now $s$ section of $\mathcal{F}'$ on $\mathcal{U}$ Zariski-open. $s \otimes 1$ section of $\mathcal{F}^h$ over $\mathcal{U}^h$.

This commutes with restriction and cobord maps $\rightsquigarrow$ $\varepsilon: H^q(X, \mathcal{F}) \to H^q(X^h, \mathcal{F}^h)$

Thm 1. $\varepsilon: H^q(X, \mathcal{F}) \to H^q(X^h, \mathcal{F}^h)$ is an isom.

Thm 2. $\mathcal{F}, \mathcal{G}$ be alg. coh. sheaves. Let $\psi: \mathcal{F}^h \to \mathcal{G}^h$ an analytic morphism. Then $\exists!$ $\varphi: \mathcal{F} \to \mathcal{G}$ s.t. $\varphi^h = \psi$

Thm 3. $\mathcal{M}$ analytical coh. sheaf on $X^h$.

Then $\exists$ $\mathcal{F}$ alg. coh. $/ X$ s.t. $\mathcal{F}^h \simeq \mathcal{M}$. $\mathcal{F}$ is unique up to isom.

Pf of Thm 1

(i) Reduction to the case $X = \mathbb{P}^r_{\mathbb{C}}$

(ii) True for $\mathcal{O}$.

(iii) " $\mathcal{O}(n)$

(iv) " $\mathcal{F}$.

(i) $X \overset{i}{\hookrightarrow} \mathbb{P}^r_{\mathbb{C}}$ embedding. $\mathcal{F}$ on $X \rightsquigarrow i_*\mathcal{F}$ on $\mathbb{P}^r_{\mathbb{C}}$ is an alg. coh. sheaf

Moreover, $\check{H}^q(X, \mathcal{F}) \simeq \check{H}^q(\mathbb{P}^r_{\mathbb{C}}, i_*\mathcal{F})$

$$H^q(X^h, \mathcal{F}^h) \simeq H^q(\mathbb{P}^r(\mathbb{C}), i_*\mathcal{F}^h) \quad ✓$$

(ii) We have $H^0(\mathbb{P}^r, \mathcal{O}) = \mathbb{C}$

$$H^0(\mathbb{P}^r(\mathbb{C}), \mathcal{H}) = \mathbb{C}$$

$$H^i(\mathbb{P}^r, \mathcal{O}) = 0 \quad \forall i > 0$$

$$H^i(\mathbb{P}^r(\mathbb{C}), \mathcal{H}) \simeq H^{0,i}(\mathbb{P}^r) = 0 \quad \forall i > 0.$$

(iii) By induction on $r = \dim \mathbb{P}^r$

$r = 0$ ✓

$E = \{t_0 = 0\} \subset \mathbb{P}^r$, $t_0, \dots, t_r$ hom. coord's

$$0 \to \mathcal{O}(m-1) \to \mathcal{O}(m) \to \mathcal{O}_E(m) \to 0 \quad \overset{\text{L.E.S.}}{\rightsquigarrow}$$

$$\begin{array}{ccccccccc} H^{i-1}(E, \mathcal{O}_E(m)) & \to & H^i(\mathbb{P}^r, \mathcal{O}(m-1)) & \to & H^i(\mathbb{P}^r, \mathcal{O}(m)) & \to & H^i(E, \mathcal{O}_E(m)) & \to & H^{i+1}(\mathbb{P}^r, \mathcal{O}(m-1)) \to \\ \downarrow\wr & & \downarrow & & \downarrow & & \downarrow\wr & & \downarrow \end{array}$$

Hence, by 5-lemma, true for $\mathcal{O}(m) \iff$ true for $\mathcal{O}(m-1)$.

We have it for $m=0$, so for all $m$.

(iv) $\mathcal{F}$ alg. coh. sheaf on $\mathbb{P}^r$.

By Serre, $\exists\, n_0$ s.t. $\forall n \geq n_0$, $\mathcal{F}(n)_x$ is gen by $H^0(\mathbb{P}^r, \mathcal{F}(n))$,

i.e. we have ~~an~~ a surjection

$$\mathcal{O}^l(-n) \to \mathcal{F} \to 0 \quad \text{on } \mathbb{P}^r.$$

If $K$ is the kernel, L.E.S. $\rightsquigarrow$

$$\begin{array}{ccccccccc} H^i(\mathbb{P}^r, K) & \to & H^i(\mathbb{P}^r, \mathcal{O}^l(-n)) & \to & H^i(\mathbb{P}^r, \mathcal{F}) & \to & H^{i+1}(\mathbb{P}^r, K) & \to & H^{i+1}(\mathbb{P}^r, \mathcal{O}^l(-n)) \\ 1\ \downarrow \varepsilon & & 2\ \downarrow \wr & & 3\ \downarrow & & 4\ \downarrow \wr & & 5\ \downarrow \wr \end{array}$$

For $i > 2r$, $H^i(\mathbb{P}^r, \mathcal{F}) = 0$ $\forall \mathcal{F}$ in analytic and alg. case.

2, 4, 5 isom $\overset{5\text{-lemma}}{\Longrightarrow}$ 3 surj.

We have a surj. 1

5 lemma $\Rightarrow$ Thm 1 $\square$

Pf Thm 2

Start with $\psi : \mathcal{F}^h \to \mathcal{G}^h$ analytic morph.

We want $\varphi : \mathcal{F} \to \mathcal{G}$, i.e. something in $H^0(X, \mathcal{H}om(\mathcal{F}, \mathcal{G}))$

So enough to show

$$H^0(X, \mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})) \simeq H^0(X^h, \mathcal{H}om_{\mathcal{H}}(\mathcal{F}^h, \mathcal{G}^h))$$

But $\mathcal{H}om_{\mathcal{O}}(\mathcal{F}, \mathcal{G})^h \simeq \mathcal{H}om_{\mathcal{H}}(\mathcal{F}^h, \mathcal{G}^h) \overset{\text{Thm 1}}{\Longrightarrow}$ Thm 2. $\square$

Pf of thm 3

~~Pf~~ (i) Enough to show for $X = \mathbb{P}^r_{\mathbb{C}}$.

Indeed, $X \xhookrightarrow{i} \mathbb{P}^r_{\mathbb{C}} \rightsquigarrow X^h \xhookrightarrow{i} \mathbb{P}^r(\mathbb{C})$ $\mathcal{M}$ analyt. coh. sheaf on $X^h$

$\rightsquigarrow i_* \mathcal{M}$ on $\mathbb{P}^r(\mathbb{C})$

Thm 3 on $\mathbb{P}^r \Rightarrow i_*\mathcal{M} \simeq \mathcal{G}^h$ for some $\mathcal{G}$ alg. sheaf on $\mathbb{P}^r$

Now $\mathcal{G} \simeq i_* \mathcal{F}$ for some $\mathcal{F}$ alg coherent sheaf on $X$. Indeed, $\forall x \notin X$ $i_*\mathcal{M}_x \simeq \mathcal{G}^h_x = 0$ so you can just pullback $\mathcal{G}$ via $i^*$.

Prop. Let $\mathcal{M}$ be an analytical coh. sheaf on $\mathbb{P}^r$

Then, $\exists\, n_0$ s.t. $\forall n \geq n_0$, $\mathcal{M}(n)_x$ is gen. by global sections $H^0(\mathbb{P}^r, \mathcal{M}(n))$

Prop + Thm (i) $\Rightarrow$ Thm 3.

$$\mathcal{H}(n)^{\ell} \longrightarrow \mathcal{M} \to 0 \quad \text{by prop}$$

Let $K$ be kernel, prop for $K$

$$\mathcal{H}(n_2)^{\ell_2} \longrightarrow K \to 0$$

$$\mathcal{H}(n_2)^{\ell_2} \xrightarrow{\psi} \mathcal{H}(n_1)^{\ell_1} \to \mathcal{M} \to 0, \quad \text{here } n_i \in \mathbb{Z}$$

Now $\mathcal{O}(n_2)^{\ell_2} \xrightarrow{\varphi} \mathcal{O}(n_1)^{\ell_1}$ by thm 2 with $\varphi^h = \psi$,

then $\operatorname{coker} \varphi =: \mathcal{F}$. $\Rightarrow \mathcal{F}^h \simeq \mathcal{M}$.

Still to prove prop:

$A_r$) $\mathcal{M}(n)_x$ gen. by $H^0(\mathbb{P}^r, \mathcal{M}(n))$

$B_r$) $H^i(\mathbb{P}^r, \mathcal{M}(n)) = 0 \quad \forall i > 0$

$A_{r-1}) + B_{r-1}) \Rightarrow A_r$

$A_r \Rightarrow B_r$

Rem. If $H^0(\mathbb{P}^r, \mathcal{M}(n))$ generates $\mathcal{M}(n)_x$, then it generates $\mathcal{M}(n)_y$ $\forall y$ ~~to a~~ close.

Rem. + $\mathbb{P}^r$ compact $\Rightarrow$ We can prove prop. $A_r$ for one $x \in \mathbb{P}^n$.

Let $E = \{t_0 = 0\} \subset \mathbb{P}^r$ ~~pa~~ containing $x$. Then

$$0 \to \mathcal{B}(n) \to \mathcal{M}(n-1) \xrightarrow{\cdot t_0} \mathcal{M}(n) \to \mathcal{M}_E(n) \to 0 \quad \rightsquigarrow \text{Break it}$$

$$0 \to \mathcal{B}(n) \to \mathcal{M}(n-1) \to \mathcal{L} \to 0$$

L.E.S. $\rightsquigarrow$

$$0 \to \mathcal{L} \to \mathcal{M}(n) \to \mathcal{M}_E(n) \to 0$$

$$H^1(\mathbb{P}^r, \mathcal{M}(n-1)) \to H^1(\mathbb{P}^r, \mathcal{L}) \to H^2(\mathbb{P}^r, \mathcal{B}(n))$$

$$H^1(\mathbb{P}^r, \mathcal{L}) \to H^1(\mathbb{P}^r, \mathcal{M}(n)) \to H^1(\mathbb{P}^n, \mathcal{M}_E(n))$$

For $n \gg 0$, by induction hyph $A_{r-1}$, we have $H^2(\mathbb{P}^r, \mathcal{B}(n)) = 0$

$H^1(\mathbb{P}^n, \mathcal{M}_E(n)) = 0$

Hence, $H^1(\mathbb{P}^r, \mathcal{M}(n-1)) \twoheadrightarrow H^1(\mathbb{P}^r, \mathcal{L})$

$H^1(\mathbb{P}^n, \mathcal{L}) \twoheadrightarrow H^1(\mathbb{P}^r, \mathcal{M}(n))$

For a compact ~~close~~ complex variety $X$ and for $\mathcal{M}$ analytic coherent sheaf, $H^q(X, \mathcal{M})$ has finite dimension. Then

$$h^1(\mathbb{P}^r, \mathcal{M}(n-1)) \geq h^1(\mathbb{P}^r, \mathcal{L}) \geq h^1(\mathbb{P}^r, \mathcal{M}(n))$$

For $n \gg 0$, they are equalities. Hence,

$$H^0(\mathbb{P}^r, \mathcal{M}(n)) \to H^0(\mathbb{P}^r, \mathcal{M}_E(n)) \xrightarrow{\;\searrow_0 \nearrow\;} H^1(\mathbb{P}^r, \mathcal{L}) \twoheadrightarrow H^1(\mathbb{P}^r, \mathcal{M}(n))$$

$$H^0(\mathbb{P}^r, \mathcal{M}(n)) \twoheadrightarrow H^0(\mathbb{P}^r, \mathcal{M}_E(n)) \simeq H^0(E, \mathcal{M}_E(n))$$

is a surjection.

By induction for $n \geq n_0$, $\mathcal{M}_E(n)_x$ is generated by $H^0(E, \mathcal{M}_E(n))$

Let $A = \mathcal{H}_x$, $I = \mathcal{I}_{E,x}$, $A/I = \mathcal{H}_{x,E}$

$M = \mathcal{M}(n)_x$, $\quad N = H^0(\mathbb{P}^r, \mathcal{M}(n)) \cdot \mathcal{M}(n)_x$

We want $M = N$ $\quad$ we have $\quad M/IM = N/IM$

because $\mathcal{M}_E(n)_x = M \otimes A/I \simeq M/IM$

$$= H^0(E, \mathcal{M}_E(n)) \cdot \mathcal{M}_E(n)_x = H^0(\mathbb{P}^r, \mathcal{M}(n)) \cdot \mathcal{M}_E(n)_x$$

Hence $M = N + IM \underset{\text{Nakayama}}{\Rightarrow} M = N$.

This gives $A_{r-1} + B_{r-1} \Rightarrow A_r$

For $A_r \Rightarrow B_r$, check Serre.

Serre was inspired by Kodaira-Spencer for the finite dimension.

$$\begin{array}{c} L \\ \downarrow \\ X \end{array} \quad \text{line bdle over } \underset{\text{compact variety}}{X}, \qquad H^1(X, L) \text{ is finite dimensional.}$$

We have $H^q(X, \Omega^p(L)) = \overset{\to\ \text{harmonic forms}}{\mathcal{H}^{p,q}(X, L)}$

$$= \{\Delta\alpha = 0\}$$

$\Delta$ is an elliptic operator

<u>Thm</u>. Over a compact manifold, an elliptic operator is Fredholm ($\Rightarrow$ kernel finite dim.).

For $\mathcal{M}$ coh. sheaf, we use $\mathcal{H}^p \to \mathcal{M} \to 0$
$\hookrightarrow$ p-holom. functions

<u>Rem.</u> For thm 2, Serre ~~introduces~~ uses of notion of flatness.

$$\begin{array}{ccccc} \mathcal{O}_{X,x} & \longrightarrow & \mathcal{H}_{X,x} & \hookrightarrow & \mathbb{C}[[z_1, \dots, z_n]] \\ & \text{flat} & \| \text{Noeth} & & \\ & & \mathbb{C}\{z_1, \dots, z_n\} & & \end{array}$$

$\mathcal{F}$ coh analytic,
$\mathcal{F} \otimes \mathcal{O}(n)$ is generated by its global sections and has larger cohom.

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## The formalism of slopes

- Vector bundles over a smooth $\underset{\text{connected}}{\frown}$ projective curve $X/\mathbb{C}$

1) Slopes and semi-stability

<u>Recall:</u>

$$\{\text{v.b. over } X\} \xleftrightarrow{\text{equiv}} \{\text{loc. free finite type } \mathcal{O}_X\text{-mod}\}$$

$$p : E \to X \longmapsto \mathcal{E} : U \mapsto \{\text{sections } s : U \to E \text{ of } p\}$$

Invariants of the $\mathcal{O}_X$-module $\mathcal{F}$

- <u>rank</u> $\to$ $rk(\mathcal{F}) := rk(\mathcal{F}|_U)$
$rk(E) := rk(\mathcal{E})$
- <u>degree</u> $\to$ $d(\mathcal{F}) := \chi(\mathcal{F}) - rk(\mathcal{F})\chi(\mathcal{O}_X)$

<u>Rem.</u> RR: if $\mathcal{F}$ is an inv. $\mathcal{O}_X$-mod, $d(\mathcal{F})$ = degree of the ass. div

- $\det(\mathcal{F}) := \Lambda^{rk \mathcal{F}} \mathcal{F} \in Pic(X)$. The degree of the divisor of $\det(\mathcal{F})$ $= d(\mathcal{F})$.

$d(E) := d(\mathcal{E})$

If $\mathrm{rk}\, \mathcal{F} > 0$, the <u>slope of $\mathcal{F}$</u> is $\mu(\mathcal{F}) := \dfrac{d(\mathcal{F})}{\mathrm{rk}(\mathcal{F})}$

$E \neq 0$ v.b. $\rightsquigarrow$ $\mu(E) := \mu(\mathcal{E})$

<u>Rem</u>. $\mu(E \otimes E') = \mu(E) + \mu(E')$

$\mu(E^*) = -\mu(E)$

$\mu(\mathcal{H}om(E, F)) = \mu(F) - \mu(E)$

<u>Lem</u> $E' \subseteq E$ non-zero subbundle (i.e. $E/E'$ also a v. bdle). Then TFAE

(i) $\mu(E') < \mu(E)$

(ii) $\mu(E') < \mu(E/E')$

(iii) $\mu(E) < \mu(E/E')$

<u>Def</u>. $E$ v.b. of rank $> 0$ and slope $\mu$. We say $E$ is <u>semi-stable</u> (resp. <u>stable</u>) if for any subbundle $E'$ of $E$ different from $0$ and $E$, we have $\mu(E') \leq \mu(E)$ (resp. $\mu(E') < \mu(E)$)

<u>Rem</u>. If $E$ is semistable, then $\forall \mathcal{E}'' \subseteq \mathcal{E}$ a coh. $\mathcal{O}_X$-submodule

$\mu(\mathcal{E}'') \leq \mu(\mathcal{E})$

Indeed, there is a subbundle $E'$ of $E$ s.t. $\mathcal{E}'' \subseteq \mathcal{E}'$, $\mathrm{rk}(\mathcal{E}'/\mathcal{E}'') = 0$

Hence, $d(\mathcal{E}') = d(\mathcal{E}'/\mathcal{E}'') + d(\mathcal{E}'') = \dim H^0(X, \mathcal{E}'/\mathcal{E}'') + d(\mathcal{E}'') \geq d(\mathcal{E}'')$

$\mathrm{rk}(\mathcal{E}') = \mathrm{rk}(\mathcal{E}'') \Rightarrow \mu(\mathcal{E}') \geq \mu(\mathcal{E}'')$

$\square$

<u>Rem</u>. Equality holds $\Leftrightarrow$ $\mathcal{E}''$ corresponds to a subbundle (i.e. $\mathcal{E}'/\mathcal{E}'' = 0$).

<u>Ex</u>. Line bdle : stable

- $X = \mathbb{P}^1$: $\mathcal{E} = \mathcal{O}_X(a_1) \oplus - \oplus \mathcal{O}_X(a_r)$

The only semistable bundles are $\mathcal{O}_X(k)^r$.

<u>Category of semistable bundles</u>

Prop. (i) $E, F$ s.s. bundles $/X$, $f : E \underset{\neq 0}{\to} F$. Then $\mu(E) \le \mu(F)$.

(ii) If $E, F$ are stable of the same slope $\mu$, then $f$ is an isom. $f : E \to F$

(iii) $E$ stable: $\mathrm{End}(E) \simeq \mathbb{C}$

Pf. (i) $\mathcal{I} = \mathrm{im}(\mathcal{E} \to \mathcal{F})$

s.s. $\Rightarrow$ $\mu(\mathcal{E}) \le \mu(\mathcal{I}) \le \mu(\mathcal{F})$

(ii) $\mu \le \mu(\mathcal{I}) \le \mu \Rightarrow \mu(\mathcal{I}) = \mu \rightsquigarrow \mathcal{I}$ corresponds to a subbundle of $\mathcal{F}$.

Stability $\Rightarrow$ $\mathcal{I} = \mathcal{F}$, $f : \mathcal{E} \xrightarrow{\sim} \mathcal{I}$

(iii) $f \in \mathrm{End}(E) \neq 0$, $\mathbb{C}[f]/\mathbb{C}$ fin. ext. $\Rightarrow \mathbb{C}[f] = \mathbb{C}$.

□

$\mathcal{C}$ = category of v.b. $/X$.

Rem. $\mathcal{C}$ additive ✓

- $\mathcal{C}$ not abelian!

↳ But if $\mu \in \mathbb{Q}$, $\mathcal{C}(\mu) :=$ full subcat. of s.s. bdles of slope $\mu$

Thm. $\mathcal{C}(\mu)$ is abelian, stable by extension.

Pf. $f : E \to F \neq 0$, $\mathcal{I} = \mathrm{im}(f)$, $\mu = \mu(\mathcal{E}) \le \mu(\mathcal{I}) \le \mu(\mathcal{F}) = \mu$

Hence $\mathcal{I}$ is a subbundle of $\mathcal{F}$.

Extension: easy

□

Filtrations

a) Jordan-Hölder filtration.

Let $E$ be a v.b. $/X$, s.s. of slope $\mu$. Then $\exists$ a stable subbundle $E_1$ of slope $\mu$ (else, strictly decreasing seq. of semistable subbundles, but rank can get smaller arbitrary!).

Start with $E/E_1$. Get: $0 \subseteq E_1 \subseteq \dots \subseteq E_k = E$ s.t.

s.t. $\forall i$, $gr_i = E_i/E_{i-1}$ stable of slope $\mu$ $\rightsquigarrow$ J.H. filtration

b) Harder-Narasimhan filtration.

<u>Prop.</u> Let $E$ be a v.b. $/X$. Then $\exists$ vector subbundles

$$0 \subseteq E_1 \subseteq \dots \subseteq E_k = E \quad \text{s.t.}$$

(i) $\forall i$, $gr_i = E_i/E_{i-1}$ semi-stable

(ii) $\forall i$, $\mu(gr_i) > \mu(gr_{i+1})$

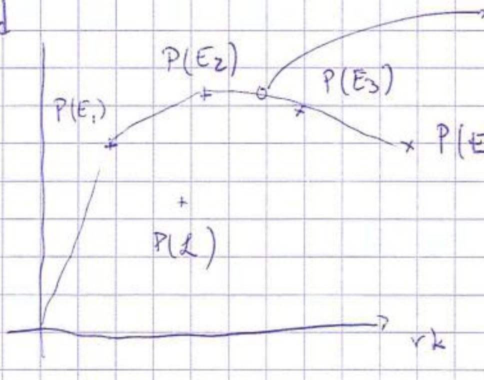
This filtration is unique.

Pf. Consider coh. sub $\mathcal{O}_X$-mod of $\mathcal{E}_1$ of maximal slope, and among, choose one of <u>maximal rank</u> → requires some argument.

$\mathcal{E}_1$ is a subbundle $E_1$.

$E/E_1$ iterate $\square$

Formalism of Newton polygons:




$P(\mathcal{L})$ on an edge $\Rightarrow$ $\mathcal{L}$ is a subbundle $E_2 \subseteq \mathcal{L} \subseteq E_3$

$P(\mathcal{L})$ inside Newton polygon.

$\mathcal{L}$ coh. sub $\mathcal{O}_X$-mod. of rank $E$

4) <u>Moduli spaces</u>

<u>Recall</u>: If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-mod. of rank $r$, we have the Hilbert polynomial:

$$P_{\mathcal{F}}(n) = \chi(\mathcal{F}(n)) = n\, r \deg(\mathcal{F}) + \chi(\mathcal{F})$$

· $P$ a degree 1 poly, $S/\mathbb{C}$ variety, $E$ v.b. $/X$

$$\underline{\mathrm{Hilb}^P(E,S)} = \left\{ \mathcal{G} \text{ coh. } \mathcal{O}_{X\times S}\text{-mod} \;\middle|\; \begin{array}{l} \text{i) } \mathcal{G} \text{ quot. of } E \\ \text{ii) } \mathcal{G} \text{ is flat} \\ \text{iii) } \forall s \in S,\ P_{\mathcal{G}(s)} = P \end{array} \right\}$$

**Thm** (Groth.) <u>Hilb</u> is repr. by a proj variety $Hilb^P(E)$

**Def.** a family $\mathcal{V}$ of v.b. over $X$ is <u>bounded</u> if $\exists$ a variety $S$ and a v.b $E$ on $S \times X$ s.t. $\mathcal{V}$ is contained in the set of v.b. classes $E(s)$, $s \in S$.

**Thm.** The set $S(r,d)$ of ss v.b. over $X$ of rk $\underset{r \geq 2\,!!!}{r}$ and deg $d$ is bounded, whereas the set $T(r,d)$ of v.b rank $r$ and deg $d$ is not.

**Pf.** . <u>$T(r,d)$</u>

<u>Fact.</u> If $\mathcal{V}$ is bounded, then $\{h^1(E) \mid E \in \mathcal{V}\}$ is finite.

But in $T(r,d)$ we have, $a \in X$

$$E = \underbrace{\mathcal{O}(-ka)}_{\deg k} \oplus \underbrace{\mathcal{O}((d-k)a)}_{\deg d-k} + \underbrace{\mathcal{O}^{r-2}}_{\deg 0}$$

$$h^1(E) \geq h^1(\mathcal{O}(-ka)) = k+g-1 \xrightarrow[k\to\infty]{} \infty \quad \Rightarrow \text{we're done}$$

- $\underset{E}{\overset{\in}{S(r,d)}}$ : $\quad \mu = \frac{d}{r}$, $\quad \gamma$ integer $> 2g-1-\mu$, $\quad a \in X$.

For $\eta \geq \gamma - 1$, then

$$\mu(E(\eta a)) = \mu(E) + \mu(\mathcal{O}(\eta a)) = \mu + \eta > 2g-2 = \mu(\omega_X)$$

$$\Longrightarrow \mathrm{Hom}(E(\eta a), \omega_X) = 0 \underset{\text{serre duality}}{\Longrightarrow} H^1(E(\eta a)) = 0$$

$H^1(E(\gamma a)) = 0$

(You have to do this $\forall a \in X$, but $\gamma$ for $a$ works in a nbhd, and $X$ compact)

$$0 \to E((\gamma-1)a) \to E(\gamma a) \to E_a \to 0$$

$$\rightsquigarrow 0 \to H^0(\quad) \to H^1(E(\gamma a)) \to H^0(E_a) \to 0$$

Hence $E(\gamma a)$ is generated by its global sections.

$H$ vect. sp. of dim $N = \chi(E(\gamma a)) = \dim H^0(E(\gamma a))$

By choosing an isom. $H \simeq H^0(E(\gamma a))$, then surj. map:

$$H \otimes \mathcal{O}(-\gamma a) \to E$$

$P$ = Hilbert polynomial of any element of $S(r,d)$.

Then the map corresponds to a pt. of $\mathrm{Hilb}^P(H \otimes \mathcal{O}(-\gamma a))$

$\Rightarrow$ bdd.

$\square$

Now let $\Omega \in \mathrm{Hilb}^P(H \otimes \mathcal{O}(-\gamma a))$ be the set of $E$ s.t. $E$ is semi-stable and $H \xrightarrow{\sim} H^0(E(\gamma a))$

<u>Cor</u>. Natural bijection

$$\Omega/_{SL(H)} \xrightarrow{\sim} S(r,d)$$

The key point is to prove $\Omega$ is the open subset given by Mumford's GIT.

Then $\Omega/_{SL(H)}$ satisfies the universal prop. for quotients, bla bla bla

$\pi$ $M(r,d) := \Omega/_{SL(H)}$

<u>Thm</u> Given $\underline{M}(r,d)$: $S \mapsto \left\{ \begin{array}{l} \text{isom. classes of v.b. } E \text{ of rk } r \text{ and deg } d \\ \text{over } X \times S \text{ s.t. } F(s) \text{ s.s. deg } d, \text{ rk } r \ \forall s \in S \end{array} \right\}$

then $M(r,d)$ is a coarse moduli space for this functor, ie

$\exists$ $\varphi : \underline{M}(r,d)(S) \to \mathrm{Mor}(S, M(r,d))$ s.t. $\forall N$ alg. var, &

$\forall$ funct. morph. $\psi : \underline{M}(r,d)(S) \to N(S)$, there $\exists!$ $f : M(r,d) \to N$ s.t.

$$\begin{tikzcd} \underline{M}(r,d)(S) \arrow[r] \arrow[dr] & M(r,d)(S) \arrow[d, dashed] \\ & N(S) \end{tikzcd}$$

There is an open subset of $M(r,d)$ which parametrizes isom. classes of stable v.b.

$S(r,d) \to M(r,d)$ is surj. and the fibers consist of v.b. which have the same ass. bundle $\oplus gr_i$ for the J.H. filtration. Thm

# Hodge index thm (HIT) for proj surfaces and connectedness thm

Let $X$ be a smooth proj surface / $k = \bar{k}$.

Intersection pairing on divisors:

$$\mathrm{Div}\, X \times \mathrm{Div}\, X \longrightarrow \mathbb{Z}$$

extends by $\otimes \mathbb{R}$ to real coeff and after quotienting by numerically trivial divisors, we get a symmetric non-deg pairing of the Neron severing $NS(X)$ (~~not~~ tensored with $\mathbb{R}$).

$$NS(X) \times NS(X) \to \mathbb{R}$$

Rem i) Thm of the base. $NS(X)$ is finitely gent., $\mathrm{rk}\, NS(X) =: \rho$.

ii) If $H \in NS(X)$ is ample and $D$ is an (irred divisor) integral curve then $H \cdot D > 0$

Also $H^2 > 0$.

Thm (HIT v1.0) $D \in NS(X)$, $\neq 0$ and $D \cdot H = 0$, then $D^2 < 0$

Fix $H$ ample div. on $X$.

Pf. Note that $H^2 > 0 \Rightarrow NS(X) = \mathbb{R}[H] \oplus H^{\perp}$, where $D \in H^{\perp}$ if $D \cdot H = 0$

So what we want to show is that $D \in H^{\perp}$, then $D^2 > 0$.

Assume $D^2 \geq 0$

RR for surfaces: $\chi(nD) = \dfrac{n^2 D^2 - nD K_X}{2} + \chi(\mathcal{O}_X)$

where $K_X$ = canonical divisor of $X$.

As a polynomial in $n$, this satisfies $\chi(nD) \to \infty$ as $n \to \infty$

By Serre duality, $h^0(nD) + h^0(K_X - nD) \geq \chi(nD)$

Since $\chi(nD) \xrightarrow[n\to\infty]{} \infty$, at least one of the terms goes to infinity

1) If $h^0(nD) \to \infty$, then $nD$ effective, so $nD \cdot H > 0 \Rightarrow D \cdot H > 0$

Also $(K_X - nD) \cdot H < 0$ for $n \gg 0$, so $h^0(K_X - nD) = 0$ $n \gg 0$.

If $h^0(K_X - nD) \to \infty$, then $K_X - nD$ eff

so $(K_X - nD).H > 0$. Hence $nD.H < 0 \Rightarrow D.H < 0 \Rightarrow h^0(nD) = 0$.

Upshot: these are mutually exclusive and in particular they give a contradiction.

Pending: $D^2 = 0$?

Thm HIT v2.0 The signature of the pairing is $(1, \rho - 1)$

HIT v3.0 $D_1, \dots, D_n$ any divisors s.t. $\exists\, \alpha_1, \dots, \alpha_n \in \mathbb{Z}$ with

$D := \sum \alpha_i D_i$ and $D^2 > 0$, then $(-1)^{n-1} \det(D_i . D_j) \geq 0$

The three versions are equivalent.

$v1.0 \Rightarrow v2.0$ —

$v3.0 \Rightarrow v1.0 \rightsquigarrow D_1 = D,\ D_2 = H$

$v2.0 \Rightarrow v3.0 \rightsquigarrow$ define $\varphi : \mathbb{R}^n \to NS(X) : (x_1, \dots, x_n) \mapsto \sum_{}^{n} x_i D_i$

If $\det(D_i . D_j) \neq 0$, $\{D_i\}$ lin. indep, so

$\varphi$ is injective. We have an induced pairing $\langle -, - \rangle$ on $\mathbb{R}^n$

given by $\langle (x_1, \dots, x_n), (y_1, \dots, y_n) \rangle := (\sum x_i D_i).(\sum y_i D_i)$

This pairing has signature $(1, n-1)$ and the positive definite part is given by $\mathbb{R}\left[\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}\right]$. The determinant of this pairing is

$\det(D_i . D_j)$ with sign $(-1)^{n-1} \Rightarrow$ —

Application: Let $C$ be a smooth proj. curve over $\overline{\mathbb{F}_p} \Rightarrow$

$\Rightarrow \exists\, q = p^r$ s.t. $C$ defined over $\mathbb{F}_q$.

Let $F : C \to C$ absolute Frobenius ($f \mapsto f^q$ on functions on $C$).

Fix $p \in C(\overline{\mathbb{F}_p})$, $V := \{p\} \times C$, $H := C \times \{p\} \subset C \times C$

$\Delta :=$ diagonal, $G :=$ graph of Frob.

Note that $G = (F, id)^{-1}(\Delta)$

If $N := \# C(\mathbb{F}_q)$, the intersection matrix is

| | $V$ | $H$ | $\Delta$ | $G$ |
|---|---|---|---|---|
| $V$ | 0 | 1 | 1 | 1 |
| $H$ | 1 | 0 | 1 | $q$ |
| $\Delta$ | 1 | 1 | $2-2g$ | $N$ |
| $G$ | 1 | $q$ | $N$ | $q(2-2g)$ |

$H . G = \deg(F) = q$

$\Delta^2 = 2-2g$ by adjunction

$\Delta \cdot G =$ fixed points of Frob $= N$

$G^2 = \deg(F) . \Delta^2$

By setting $D_1 = V$, $D_2 = H$, $D_3 = \Delta$, $D_4 = G$

$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 1, 0, 0)$

$D = \sum \alpha_i D_i = V + H$ satisfies $D^2 > 0$, so that HIT $\Rightarrow$ 3.0

the determinant of the above has sign $(-1)^{4-1} = -1$. Compute

the det $= (N-(q+1))^2 - 4g^2 q \Rightarrow$

$\Rightarrow$ Thm (Weil) $|N-(q+1)| \leq 2g\sqrt{q}$.

# Chow connected and density thm

$X$ quasi-proj. scheme$/\mathbb{C}$, wlog red., irred.

<u>Prop 1</u>. If $\mathcal{U} \subset X$ Zariski open (dense), then
$\mathcal{U}(\mathbb{C}) \subset X(\mathbb{C})$ analytically dense.

<u>Pf</u>. Reduce by hyperplanes to curve case, there normalize and use same result for Riemann surfaces.

□

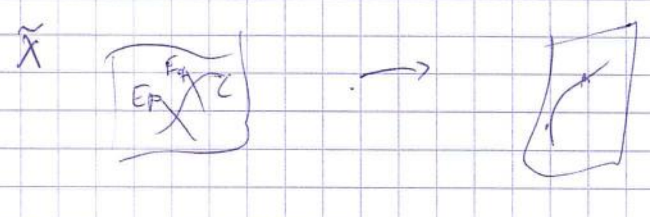
<u>Prop 2</u> $X$ Zariski connected, then $X(\mathbb{C})$ analytically connected.

<u>Pf</u>. If $X = C$ integral curve, then $\nu: \tilde{C} \to C$ normalization, and compactify $\rightsquigarrow \hat{C} \supset \tilde{C}$. Observe $\hat{C}(\mathbb{C})$ conn. $\Rightarrow C(\mathbb{C})$ conn.
Hence might as well assume $C$ sm. proj.
Take $p, q \in C(\mathbb{C})$ and $0 \neq f \in H^0(\mathcal{O}(Np - q))$ for $N \gg 0$.
Then $\underbrace{f|_{C(\mathbb{C}) \setminus \{p\}}}_{=: \tilde{f}}$ is holomorphic and $\tilde{f}(q) = 0$.

If $C = C_1 \sqcup C_2$ disc. union of 2 curves s.t. $p \in C_1$, $q \in C_2$, then $f|_{C_2}$ is holom. and since $f|(q) = 0$, $f = 0$ on $C_2$. ↯.

□

Any $X$: take $p, q \in X(\mathbb{C})$, let $\pi: \tilde{X} \to X$ be the blow up at $p, q$ and exc. divisors $E_p$, $E_q$.
$i: \tilde{X} \hookrightarrow \mathbb{P}^N_{\mathbb{C}}$. By Bertini, $\exists H_1, \dots, H_{n-1}$ general hyperplanes s.t. $X \cap \bigcap H_i = C$ irr. curve.
Moreover $C.E_p > 0$ and $C.E_q > 0$




$\rightsquigarrow$ reduced to curves

Thm (Bertini) $X$ irred. variety, $f: X \to \mathbb{P}^r$ gen. finite onto its image.
For $d < \dim X$, if $L \subset \mathbb{P}^r$ a general $(r-d)$-plane, then $f^{-1}(L)$ irreducible.

Thm (Fulton - Hansen) $X$ irred. proj., $f: X \to \mathbb{P}^r \times \mathbb{P}^r$ s.t. $\dim f(X) > r$. Then $f^{-1}(\Delta_{\mathbb{P}^r}) \subseteq X$ is connected.

Cor. $X \subseteq \mathbb{P}^r$ irred. of $\dim X = n$ s.t. $2n > r$.
Then $X$ is algebraically simply connected, i.e. no non-trivial conn. étale covers

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Formal geometry

1- Basic definitions → Here everything is locally noetherian

Local picture: $A$ is an adic ring if it is complete and separated for the $I$-adic topology, $I$ ideal, i.e. $\boxed{A = \lim_n A/I^{n+1}}$

Eg: $\mathbb{Z}_p$, $k[[t]]$

⇝ Topologically ringed space: $\mathrm{Spf}(A)$ → top. space $|\mathrm{Spec}\, A/I|$
↳ sheaf of top $\mathcal{O}_{\mathrm{Spf}(A)} = \lim_n \mathcal{O}_{\mathrm{Spec}(A/I^{n+1})}$

What we have:

$$\mathrm{Spf}(A) = \mathrm{colim}\left(A/I^{n+1}\right) \quad \text{colimit of } \underset{\text{top.}}{\wedge}\text{ringed space}$$

c.f. Grothendieck or Illusie, not Hartshorne

Ex: i) $A$ any (noeth.) ring, $I = 0$ ⇝ Affine schemes

ii) $A = \mathbb{Z}_p$, $I = (p)$, $|\mathrm{Spf}(\mathbb{Z}_p)| = |\mathrm{Spec}\, \mathbb{F}_p|$
+ sheaf $\mathbb{Z}_p$ with its top.

Now we want to glue.

A (loc. noeth.) formal scheme is a top. ring. sp. $(X, \mathcal{O}_X)$ locally isom. to some $\mathrm{Spf}(A)$.

They form a category with morphisms of ringed spaces + continuous + local

<u>Basic construction</u>: $X$ = loc. noeth. sch.

$Z = V(I) \subseteq X$ closed subscheme defined by $I \subseteq \mathcal{O}_X$

$\hat{X} = X_{/Z} = \operatorname{colim}_n X_n$,

$X_n = \mathrm{Spec}\left(\mathcal{O}_X / I^{n+1}\right)$ $\rightsquigarrow$ <u>Formal completion of $X$ along $Z$</u>

$\downarrow$

$X_{n+1}$ nilpotent thickening

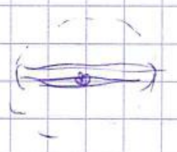
<u>Formal germs</u>: $k$ = field of char. $0$ $\quad \hat{X} = X_{/Z}$, $X, Z/k$.

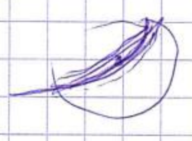
A <u>smooth formal germ</u> of $X$ through $Z$ is $\mathcal{V} \subseteq \hat{X}$ closed formal subscheme s.t.

1) $\mathcal{V}$ is formally smooth
2) $Z \subset \mathcal{V}$ + same top. space.

$X = \mathbb{A}^2_k$, $Z = (0,0)$, $\hat{X} = \mathrm{Spf}\ k[[x,y]]$ 

$\mathcal{V} = \mathrm{Spf}\ k[[x]]$, $\quad \mathcal{V}' = \mathrm{Spf}\left(k[[x,y]] / (y+1) - \exp(x)\right)$





## 5. <u>Group schemes</u>

$G$ = smooth alg. gp. scheme $/k$ $\rightsquigarrow$ $\mathfrak{g}$ = Lie algebra of $G$.

whose functor of points is $\mathfrak{g}(R) = \ker\left(G\left(R[\varepsilon]/\varepsilon^2\right) \to G(R)\right)$

$\varepsilon \longmapsto 0$

$\rightsquigarrow \hat{\mathfrak{g}}$ = completion of $\mathfrak{g}$ along zero (section)

$\hat{G}$ = completion of $G$ along $e$

Prop.
$$\begin{array}{ccc} \hat{\mathfrak{g}} & & \text{Exists ! an} \\ \downarrow \widehat{\exp}_G & & \text{formal exponential is a canonical isomorphism} \\ \hat{G} & & \text{of formal schemes (not gp schemes)} \end{array}$$

such that

(i) The diff at zero is the identity

(ii) $\forall$
$$\begin{tikzcd} \widehat{\mathbb{G}_a} \arrow[r, "\phi"] \arrow[dr, dashed] & \hat{\mathfrak{g}} \arrow[d, "\widehat{\exp_G}"] \\ & G \end{tikzcd}$$
, $\widehat{\exp_G} \circ \phi$ is a gp hom., $\forall \phi$ vector gp morphism

$\widehat{\mathbb{G}_a}$ = completion of $\mathbb{G}_a$ along zero

$\widehat{\mathbb{G}_a}(R)$ = Nilpotent elements of $R =: Nil(R)$

Pf: Uses Hopf algebra stuff + more things, not too complicated.

Example: $G = GL_n$

$$\mathfrak{g}(R) \cong M_n(R) \quad \rightsquigarrow \quad \hat{\mathfrak{g}}(R) \simeq M_n(Nil(R))$$
$$\Big\downarrow \exp$$
$$\widehat{G(R)} \simeq I + M_n(Nil(R))$$

The Grothendieck comparison thm

1- Mittag-Leffler: $(A_n, \phi_{n',n})$ inverse system of ab. gps. It satisfies

ML if $\forall n \in \mathbb{N}$, $\exists n_0 \geq n$ s.t. $\forall n', n'' \geq n_0$,

$$\begin{array}{l} A_{n'} \xrightarrow{\phi_{n',n}} A_n \\ A_{n''} \xrightarrow{\phi_{n'',n}} A_n \end{array}$$
have same images.

**Lemma** If $(A_n, \phi_{n',n})$ s.t. ML $\Rightarrow$ $R^q \lim\limits_n A_n = 0 \quad \forall q > 0$

**Artin-Rees Lemma** $A$ = noeth ring, $I \subset A$ id., $M$ = fin. gen. $A$-mod, and $M' \subset M$ sub-$A$-mod $\Rightarrow$

$\Rightarrow \exists n_0 > 0$ s.t. $\forall n \geq n_0$,

$$M' \cap I^n M = I^{n-n_0}(M' \cap I^{n_0} M)$$

In particular,

1) The $I$-adic topology on $M$ induces " on $M'$

2) The $I$-adic completion is exact on fin. gen. $A$-modules.

3) Canonical isom $\hat{A} \otimes_A M \simeq \hat{M}$

**Setting** $f: X \to Y$ morphism of loc. noeth. schemes, $Y_0 \hookrightarrow Y$,

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ Y_0 \arrow[r, hook] & Y \end{tikzcd} \quad \overset{\text{completions}}{\longmapsto} \quad \begin{tikzcd} \operatorname{colim} X_n = \hat{X} \arrow[r, hook, "i"] \arrow[d] & X \arrow[d] \\ \operatorname{colim} Y_n = \hat{Y} \arrow[r, hook, "j"] & Y \end{tikzcd} \quad \mathcal{F} = \mathcal{O}_X\text{-mod}$$

- **Base change map**: $\gamma: j^* R^q f_*(\mathcal{F}) \to R^q \hat{f}_*(i^* \mathcal{F})$

**Thm.** Assume i) $\mathcal{F}$ is coherent. Then $i^* \mathcal{F} = \lim \mathcal{F} \otimes \mathcal{O}_{X_n} = \hat{\mathcal{F}}$

ii) $f$ of finite type, and $\operatorname{supp}(\mathcal{F})$ proper/$Y$. Then base change

$$\gamma: \widehat{R^q f_* \mathcal{F}} \longrightarrow R^q \hat{f}_* \hat{\mathcal{F}}$$

is an $\mathcal{O}_{\hat{Y}}$-isom.

**Sketch.** $f$ proper, $Y = \operatorname{Spec}(A)$ $A$ ring, $\hat{Y} = \operatorname{Spf}(\hat{A})$

We reduce to prove the following two isom.:

$$\begin{tikzcd} H^q(\hat{X}, \hat{\mathcal{F}}) \arrow[r, "a", "\sim"'] & \lim_n H^q(X, \mathcal{F}/I^{n+1}) \\ & \lim_n H^q(X, \mathcal{F})/I^{n+1} \arrow[u, "b", "\wr"'] \arrow[ul, dashed, "\gamma"] \end{tikzcd}$$

a = continuity

b = completion

<u>Application</u>: $f$ proper, $Y_0 = y$ point.

Then $\widehat{\left(R^q f_* \mathcal{O}_X\right)_y} \simeq \lim_n H^q(X_y, \mathcal{O}_X/m_y^{n+1})$

$\rightsquigarrow$ This gives somehow connectedness thm.

<u>Cor</u>. If $\mathcal{O}_Y = f_* \mathcal{O}_X$, then the fibers of $f$ are non-empty and connected.

<u>Algebraization problem</u>

Let $A$ be an adic noeth. ring with ideal $I$.

$$\begin{tikzcd} & \hat{Y} = \mathrm{Spf}(A) \arrow[dl] & \\ Y = \mathrm{Spec}\, A & & \mathrm{Spec}\, A/I = Y_0 \arrow[ll, hookleftarrow] \arrow[ul] \end{tikzcd}$$

Let $\mathcal{Z}$ be an adic noeth. formal scheme $/ \hat{Y} = \operatorname{colim}_n \underbrace{(\mathcal{Z} \times_{\hat{Y}} Y_n)}_{\text{scheme}}$

Thm. Same notation for $Y$, $f: X \to Y$ of fin. type $\Rightarrow$

$\Rightarrow$ we have an equiv. of categories

$$\left\{ \begin{array}{c} \mathcal{F} \text{ coherent } \mathcal{O}_X\text{-mod} \\ \operatorname{supp} \mathcal{F} \text{ proper } / Y \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \mathcal{G} \text{ coherent } / \hat{X} \\ \operatorname{supp} \mathcal{G} \text{ proper } / \hat{Y} \end{array} \right\}$$

(Existence theorem of Groth.)

## Andreotti and Hartshorne

(1) An algebraization result.

Let $V$ be a complex manifold, $M \subset V$ a submanifold. We define the normal bundle to $M$ in $V$, a vb on $M$ given by

$$0 \to TM \to TV|_M \to N_V M \to 0$$

$\hookrightarrow rk = \dim V - \dim M$

If $M$ is a closed submanifold, $I \subset \mathcal{O}_V$ giving it, then
$N_V M \cong \left( I/I^2 |_M \right)^*$

**Def** A submanifold $V \subset \mathbb{P}^N(\mathbb{C})$ is <u>algebraizable</u> if
$\dim \left( \overline{V}^{zar} \right) = \dim(V)$.

<u>Rem</u>. Recall that we had $\geq$.

<u>Eg</u>: 1) $V = \{(z, z\sqrt{1+z}), z \in \mathbb{C}, |z| < 1\}$, $\overline{V}^{zar} = \{ y^2 = x^3 + x^2 \}$

2) $V' = \{(z, e^z), z \in \mathbb{C}\}$ $\overline{V'}^{zar} = \mathbb{P}^2(\mathbb{C})$

<u>Thm</u> 1. $X \subseteq \mathbb{P}^N(\mathbb{C})$ smooth proj. connected subvariety, $\dim X \geq 1$.
Let $V \subseteq \mathbb{P}^N(\mathbb{C})$ a complex analytic submanifold (not nec closed) s.t. $X \subseteq V$, $V$ connected.
If <u>$N_X V$ is ample</u>, $V$ is algebraizable.

<u>Ample vector bundles</u>

Let $X$ be a proj. scheme over $k$. $R_{\underline{q}}$: $H \subseteq X$ hypersurface
Divisor $\mathcal{O}(H) \simeq N_H X$

<u>Def</u> $Y \subseteq X$ <u>subvariety is ample</u> if "$N_Y X$ is ample".

<u>Recall</u>: $L$ line bundle $/X$. TFAE

1) $\exists\, m > 0$. $L^m$ is very ample $X \xrightarrow{i^m} \mathbb{P}^N$

2) $\forall \mathcal{F} \in Coh(X)$, $\exists n_0 > 0$ : $\forall m \geq n_0$, $\mathcal{F} \otimes L^m$ is globally gen.

3) — " — $\forall q > 0$, $\exists n_0$ $\forall m \geq n_0$, $H^q(X, \mathcal{F} \otimes L^m) = 0$

Now we fix $E \to X$ v.b. of rank $r+1$. We associate a proj. bundle $\mathbb{P}(E) := Proj(\underbrace{Sym\, \mathcal{E}}_{\bigoplus_{n \geq 0} S^n E})$, $S^n E = \overset{n}{\otimes} E / \text{permutations}$

$\pi : \mathbb{P}(E) \to X$, $\mathcal{O}_{\mathbb{P}(E)}(1)$ <u>Tautological line bdle</u>.

If $E$ is trivial, $E \simeq X \times \mathbb{A}^{r+1} \leftrightarrow \mathcal{E} \simeq \overset{r+1}{\bigoplus} \mathcal{O}_X$

$\mathbb{P}(E) = \mathbb{P}^r_X$ $\quad \mathcal{O}_{\mathbb{P}(E)}(1) = \mathcal{O}_{\mathbb{P}^r_X}(1)$

<u>Rem</u>: There are two conventions, $\mathbb{P}(E)$ and $\mathbb{P}(E^\vee)$. Here we follow Grothendieck.

<u>Def</u>. $E \to X$ is <u>ample</u> if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample on $\mathbb{P}(E)$.

<u>Prop</u>. $E \to X$ v.b., $n \in \mathbb{N}$

1) $\pi_*(\mathcal{O}_{\mathbb{P}(E)}(n)) \simeq S^n \mathcal{E}$

2) $\forall q > 0$, $R^q \pi_*(\mathcal{O}_{\mathbb{P}(E)}(n)) = 0$

3) $\forall q \geq 0$, $H^q(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(n)) \simeq H^q(X, S^n \mathcal{E})$

<u>Pf</u>. $X = Spec\, A$, $\mathcal{E} \simeq \mathcal{O}_X^{\oplus r+1}$

1) $H^0(\mathbb{P}^r_X, \mathcal{O}_{\mathbb{P}^r_X}(n)) \simeq A[T_0, \dots, T_r]_n$

2) $H^q(\mathbb{P}^r_X, \mathcal{O}_{\mathbb{P}^r_X}(n)) = 0$, $q > 0$, $n \geq 0$

1), 2) $\Rightarrow$ 3) by spectral seq.

$\square$

<u>Prop</u>. $E \to X$ v.b., TFAE

1) $E$ ample

2) $\forall \mathcal{F} \in Coh(X)$, $\exists n_0$, $\forall n \geq n_0$ $S^n \mathcal{E} \otimes \mathcal{F}$ is glob. gen

3) $\forall \mathcal{F} \in Coh(X), \ \forall q > 0, \ \exists n_0, \ \forall n \geq n_0, \ H^q(X, S^n \mathcal{E} \otimes \mathcal{F}) = 0$

(3) Pf of thm 1

$X \subset \mathbb{P}^N(\mathbb{C})$ sm proj subvariety, $\dim X > 0$, $X \subset V$.

$V$ submanifold, connected of $\mathbb{P}^N(\mathbb{C})$, $N_X V$ is ample $\Rightarrow$ $V$ is algebr.

Rem GAGA $N_X V$ analytic $\Rightarrow$ algebraic, so we understand "ample"

$L := \mathcal{O}_{\mathbb{P}^N_{\mathbb{C}}}(1)\big|_V$, $\quad D \in \mathbb{N}$

$$\eta_D : H^0(\overline{V}^{Zar}, \mathcal{O}(D)) \hookrightarrow H^0(V, \underset{\mathcal{O}(1)^D|_V}{\underset{\shortparallel}{L^D}})$$

is injective, (otherwise $s \in \ker(\eta_D) \setminus \{0\}$, $V \subseteq \overline{V}^{Zar} \cap \{s = 0\} \subsetneq \overline{V}^{Zar}$ ↯)

<u>Claim 1</u>: $\dim H^0(V, L^D) = \mathcal{O}(D^{\dim V})$

$$\begin{array}{c} \dim H^0(V, L^D) = \mathcal{O}(D^{\dim V}) \\ \geq \\ \dim H^0(\overline{V}^{Zar}, \mathcal{O}(D)) \\ \geq \\ C \cdot D^{\dim \overline{V}^{Zar}} \end{array}$$

$\exists c > 0$: $\Rightarrow \dim V \geq \dim \overline{V}^{Zar}$ $\rightsquigarrow$ and we are done!

Left to show: claim 1:

$I \subset \mathcal{O}_V$ ideal sheaf of $X$. $\quad \mathcal{O}_V \supset I \supset I^2 \supset \cdots$

$L^D \supset I \otimes L^D \supset \cdots$

$\rightsquigarrow$ Filtration $E^i_D := H^0(V, L^D \otimes I^i) = \left\{ \begin{array}{l} s \in H^0(V, L^D) \text{ vanish along } X \\ \text{with order} \geq i \end{array} \right\}$

$\rightsquigarrow H^0(V, L^D) = E^0_D \supset E^1_D \cdots$

and $\bigcap_{i \geq 0} E^i_D = \{0\} \Rightarrow \dim H^0(V, L^D) = \sum_{i \geq 0} \dim \left( E^i_D / E^{i+1}_D \right)$

$\rightsquigarrow 0 \to I^{i+1} \to I^i \to I^i / I^{i+1} \to 0 \quad \Big\} \ \Gamma( \ \cdot \ \otimes L)$

$$0 \to H^0(V, L^D \otimes I^{i+1}) \to \underset{E^i_D}{\underset{\shortparallel}{H^0(V, L^D \otimes I^i)}} \to H^0(V, L^D \otimes I^i / I^{i+1}) \to \cdots$$

Now $E_D^i / E_D^{i+1} \subset H^0(V, L^D \otimes I^i/I^{i+1}) \simeq H^0(X, L^D \otimes I^i/I^{i+1})$

since $\operatorname{sup} I^i/I^{i+1} \subseteq X$.

(Hence $(I^i/I^{i+1}) \simeq j_* S^i(N_X V^*)$ $\quad j : X \hookrightarrow V$)

$$\simeq H^0(X, L^D \otimes S^i(N_X V^*)) \underset{\text{char } 0}{=} H^0(X, L^D \otimes S^i(N_X V)^*) \simeq$$

$d = \dim X > 0$

$$\simeq H^d(X, L^{-D} \otimes S^i(N_X V) \otimes \omega) \simeq H^d(\mathbb{P}(N_X V), \mathcal{O}_{\mathbb{P}(N_X V)}(i) \otimes \pi^* L^{-D} \otimes \pi^*\omega)$$

Claim 2: $A$ ample line bdle $/X$, $M$ line bdle on $X$, $F \in \mathrm{Coh}(X)$,

$q > 0$. ① $H^q(X, A^i \otimes M^D \otimes F) = 0$
$\exists e > 0$ s.t. $\forall i \geq eD$

② $\exists \alpha, \beta > 0$ s.t. $\forall i, D \geq 0$, $\dim(H^q(X, A^i \otimes M^D \otimes F)) \leq \alpha (i+D)^{\dim X} + \beta$

Pf: $\exists m$, $A^m \otimes M$ ample, $e = m+1 \rightsquigarrow i \geq mD + D$, $H^q(X, (A^m \otimes M)^D \otimes A \otimes F)$

Conclusion:

$$\dim(H^0(V, L^D)) \leq \sum_{i \geq 0} \dim H^0(X, L^D \otimes I^i/I^{i+1}) =$$

$$= \sum_{i \geq 0} \dim H^d(\mathbb{P}(N_X V), \mathcal{O}_{\mathbb{P}(N_X V)}(i) \otimes (\pi^* L^{-1})^D \otimes \pi^* \omega)$$

$$\leq \sum_{i=0}^{\lceil eD \rceil} \alpha (i+D)^{\dim \mathbb{P}(N_X V)} + \beta = O(D^{\dim \mathbb{P}(N_X V) + 1})$$

$$= O(D^{\dim V})$$

④ Hartshorne's thm

Thm: Let $X \subseteq \mathbb{P}_k^N$ sm proj. subv, $\dim X > 0$,

$\hat{V} \hookrightarrow \widehat{\mathbb{P}_{k/X}^N}$ s.t. $|\hat{V}| = |X|$

If $N_X \hat{V}$ ample $\Rightarrow \hat{V}$ algebraizable formal scheme

Thm ([Har 6.7 [Har 68]): Under above assumptions, $K(\hat{V})$ has tr deg $\leq \dim \hat{V}$

field of merom. function

Rem. - Slope formalism

$$H^0(X, L^{\otimes d} \otimes S^i N^\vee)$$

If $\mu_{max} < 0$, there are no sections

$$\mu_{max}(L \otimes V) = \deg L + \mu_{max}(V)$$

$$\mu_{max}(S^i V) = i\, \mu_{max}(V)$$

Then $\mu_{max}(L^{\otimes d} \otimes S^i N^\vee) = d \cdot \deg L + i\, \mu_{max}(N^\vee)$

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SGA 2

(I) Setting.

Let $k$ be a field, $X \subseteq \mathbb{P}^N_k$ smooth proj. of pure dim. $d$

Let $t \in H^0(\mathbb{P}^N_k, \mathcal{O}(1))$ that intersects $X$ properly, $\{t = 0\} =: Y \subset X$ has dim. $d-1$

We want to compare $Y$ and $X$.

Introduce $Y_n = \{t^{n+1} = 0\} \subseteq X \rightsquigarrow \hat{X}_Y = \varinjlim Y_n$

Strategy: $Y \xrightarrow{\text{compare with}} \hat{X}_Y \xrightarrow{\text{compare}}$ some open $Y \subseteq U \subseteq X \longrightarrow X$

deformation problem — algebraization problem — Extension

(II) Main theorems

Thm (Comparison) Let $E \to X$ be a vector bdle. If $d \geq 2$,

$$H^0(X, E) \xrightarrow{\sim} H^0(\hat{X}_Y, E|_{\hat{X}_Y})$$

**Thm** (Existence). Let $\mathcal{E}$ be a v.b. on $\widehat{X}_Y$ ($E_n \to Y_n$ v.b., $E_n|_{Y_{n-1}} \simeq E_{n-1}$)

If $d \geq 3$, there exists a coherent sheaf $E$ on $X$ s.t. $E|_{\widehat{X}_Y} = \mathcal{E}$.

**Rem**. If $d=1$, thm 1 fails:

$X = \mathbb{P}^1_k$, $Y = pt$, $E = \mathcal{O}_X$

$H^0(X, \mathcal{O}_X) = k$

$H^0(\widehat{X}_Y, \mathcal{O}_{\widehat{X}_Y}) = k[[t]]$

**Rem**. Thm 2 fails if $d = 2$ $\rightsquigarrow$

$\rightsquigarrow$ $X = \mathbb{P}^2_k$, $Y = \mathbb{P}^1_k \underset{\text{line}}{\hookrightarrow} \mathbb{P}^2_k$, there are line bdles on $\widehat{X}_Y$ that do not algebraize.

**Rem** In SGA 2, we have

* Weaker hypotheses on singularities of $X$
* variants over a base
* local variants

(III) Comparison thm

**Pf**: $0 \to \mathcal{O}_X(-n-1) \to \mathcal{O}_X \to \mathcal{O}_{Y_n} \to 0$

$\downarrow \otimes E$

$0 \to E(-n-1) \to E \to E|_{Y_n} \to 0$ $\rightsquigarrow$

$$\rightsquigarrow \quad 0 \to H^0(X, E(-n-1)) \to H^0(X,E) \to H^0(Y_n, E) \to H^1(X, E(-n-1)) \to \cdots$$

Serre duality $\|$ $H^d(X, E^\vee \otimes \omega_X(n+1))^\vee$ $\|$ $n \gg 0$ $0$

$\|$ Serre $H^{d-1}(X, E^\vee \otimes \omega_X(n+1))^\vee$ $\|$ Serre vanishing, $n \gg 0$, since $d \geq 2$ $0$

Thus $H^0(X,E) \xrightarrow{\sim} \varprojlim_n H^0(Y_n, E) = H^0(\widehat{X}_Y, E|_{\widehat{X}_Y})$

(IV) Existence thm ($d \geq 3$)

Pf.

Lemma Let $\mathcal{E}$ be a vb on $\widehat{X_Y}$. Then

(i) $H^i(\widehat{X_Y}, \mathcal{E})$ is finite dimensional for $i = 0, 1$.

(ii) If $l \gg 0$, $\underbrace{H^0(\widehat{X_Y}, \mathcal{E}(l)) \otimes \mathcal{O}_{\widehat{X_Y}} \twoheadrightarrow \mathcal{E}(l)}$

$\mathcal{E}(l)$ is globally gen.

End of pf: by lemma, $\exists\, l, r$ s.t.

$$\mathcal{O}_{\widehat{X_Y}}(-l)^{\oplus r} \twoheadrightarrow \mathcal{E} \to 0$$

Apply this again to $\ker(\twoheadrightarrow)$ $\leadsto$ we obtain an exact seq

$$\mathcal{O}_{\widehat{X_Y}}(-l')^{\oplus r'} \xrightarrow[\hat{\phi}]{} \mathcal{O}_{\widehat{X_Y}}(-l)^{\oplus r} \to \mathcal{E} \to 0$$

By thm 1, $\hat{\phi} \in H^0(\widehat{X_Y}, \mathcal{H}om(\mathcal{O}(-l')^{\oplus r'}, \mathcal{O}(-l)^{\oplus r}))$ comes from

$$\phi \in H^0(X, \mathcal{H}om(\mathcal{O}(-l')^{\oplus r'}, \mathcal{O}(-l)^{\oplus r}))$$

Define $E := \operatorname{coker}(\phi)$ and you are done.

Pf of lemma:

(i)

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & \mathcal{O}_X(-n-1) \arrow[r, "t"] \arrow[d, "t^{n+1}"] & \mathcal{O}_X(-n) \arrow[r] \arrow[d, "t^n"] & \mathcal{O}_Y(-n) \arrow[r] & 0 \\
 & \mathcal{O}_X \arrow[r, equal] \arrow[d] & \mathcal{O}_X \arrow[d] & & \\
0 \arrow[r] & \mathcal{O}_Y(-n) \arrow[r, dashed] & \mathcal{O}_{Y_n} \arrow[r] \arrow[d] & \mathcal{O}_{Y_{n-1}} \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

Snake lemma

$\xrightarrow{\otimes \mathcal{E}\ +\ \text{cohom}}$

$$0 \to H^0(Y, \mathcal{E}_0(-n)) \to H^0(Y_n, \mathcal{E}|_{Y_n}) \to H^0(Y_{n-1}, \mathcal{E}|_{Y_{n-1}}) \to$$

$$\to H^1(Y, \mathcal{E}_0(-n)) \longrightarrow$$

$$" H^{d-1}(Y, \mathcal{E}_0^\vee \otimes \omega_Y(n))^\vee \underset{n \gg 0}{=} 0 \qquad = H^{d-2}(Y, \mathcal{E}_0^\vee \otimes \omega_Y(n)^\vee) \underset{n \gg 0}{=} 0$$

since $d \geq 3$

Then $H^0(Y_n, \mathcal{E}|_{Y_n})$

$$H^0(\hat{X}_Y, \mathcal{E}) = \underbrace{\varprojlim H^0(Y_n, \mathcal{E}_n)}$$

$$\downarrow \wr \ n_0 \gg 0$$

$H^0(Y_{n_0}, \mathcal{E}_{n_0})$, which is of fin. dim.

Now

$$H^1(\hat{X}_Y, \mathcal{E}) \xrightarrow{\sim} \varprojlim_n H^1(Y_n, \mathcal{E}_n)$$

$$\downarrow n_0 \gg 0$$

Because of Mittag-Leffler cond. $\qquad H^1(Y_{n_0}, \mathcal{E}_{n_0})$ fin. dim.

by finite dimensionality of the $H^1(Y_n, \mathcal{E}_n)$

Part (ii)

$$0 \to \mathcal{O}_X(-1) \xrightarrow{t} \mathcal{O}_X \to \mathcal{O}_Y \to 0$$

$$\int \otimes \mathcal{O}_{\hat{X}_Y} \otimes \mathcal{E}(l) \ + \text{Cohom.}$$

$$H^0(\hat{X}_Y, \mathcal{E}(l)) \to H^0(Y, \mathcal{E}(l)) \to H^1(\hat{X}_Y, \mathcal{E}(l-1)) \xrightarrow{t} H^1(\hat{X}_Y, \mathcal{E}(l)) \to H^1(Y, \mathcal{E}(l))$$

we don't expect?
this to be zero, so
we try to see if this is inj.

Trick: instead of ~~one~~ large $l \gg 0$, look at all of them together.

$$M := \bigoplus_{l \geq 0} H^1(\hat{X}_Y, \mathcal{E}(l))$$

viewed as a graded $k[t]$-module.

We get

$$\underbrace{H^1(\hat{X}_Y, \mathcal{E}(-1))}_{\text{fin. dim}} \oplus M \longrightarrow M \longrightarrow \underbrace{\bigoplus_{l \geq 0} H^1(Y, \mathcal{E}(l))}_{\text{fin. dim. by Serre vanishing}}$$

Thus $M/tM$ fin. dim. $\Rightarrow$ $M$ is a $k[t]$-module of finite type (lift a basis of $M/tM$ to $M$).

Now $N = \{x \in M \mid tx = 0\} \subseteq M$ is a $k[t]$-module of finite type $\Rightarrow$

$\Rightarrow$ $N$ has finite dim.

Hence, $N$ vanishes in high degree for the grading

Rem. This comes from Kodaira-Spencer.

Rem ~~Hence~~, $\cdot t$ is injective for $l \gg 0$.

Better way: $\cdot t$ are onto when $l \gg 0$, hence $h^1(\hat{X}_Y, \mathcal{E}(l))$ eventually decreases, so they become stationary $\Rightarrow$ injectivity

If $l \gg 0$,

$$H^0(\hat{X}_Y, \mathcal{E}(l)) \twoheadrightarrow H^0(Y, \mathcal{E}(l))$$

$\mathcal{E}(l)$ globally generated on $Y$

$\rightsquigarrow$ $H^0(\hat{X}_Y, \mathcal{E}(l)) \otimes \mathcal{O}_{\hat{X}_Y} \twoheadrightarrow \mathcal{E}_0(l)$

By Nakayama, $H^0(\hat{X}_Y, \mathcal{E}(l)) \otimes \mathcal{O}_{\hat{X}_Y} \twoheadrightarrow \mathcal{E}(l)$ $\square$

(V) Application to $\pi_1$

Zariski-Nagata purity thm

**Thm** If $d \geq 3$, $\pi_1^{ét}(Y) \simeq \pi_1^{ét}(X)$ — hyperplane section

Pf. We need to show that

$$\acute{E}t(X) \to \acute{E}t(Y) \text{ is an equiv. (finite étale covers)}$$

$$\acute{E}t(X) \xrightarrow{a)} \lim_{\substack{Y \subseteq U \subseteq X \\ op.}} \acute{E}t(U) \xrightarrow{b)} \acute{E}t(\hat{X}_Y) \xrightarrow{c)} \acute{E}t(Y)$$

a) is an equiv: fin. ét. cover of $U$ extend uniquely to a normal finite cover of $X$. Zariski-Nagata purity theorem $\rightsquigarrow$ the locus where it is not étale is a divisor, that ~~can't~~ ~~meet~~ $Y$. Hence is empty because $Y$ is ample.

b) A finite étale cover is the data of a v.b. $E$ + $\mathcal{O}_X \to E$ and mult. map $E \otimes E \to E$ with some compatibilities.

By thm 1 and 2, we get an equiv.

$$\varinjlim_{Y \subseteq U \subseteq X} (\text{vect. bdls on } U) \xrightarrow{\sim} \text{v.b.}(\hat{X}_Y)$$

This formally implies, ~~that~~ using that étale is an open condition, that

$$\lim \acute{E}t(U) \xrightarrow{\sim} \acute{E}t(\hat{X}_Y)$$

c) is an equivalence because $\acute{E}t(Y_n) \xrightarrow{\sim} \acute{E}t(Y_{n-1})$

(étale covers are insensitive to nilpotents!)

Rem. If $d \geq 2$, this proves that $\pi_1(Y) \twoheadrightarrow \pi_1(X)$

## VI Application to Pic

<u>Thm</u> $d \geq 3$, ~~$H^i(X$~~ $H^i(Y, \mathcal{O}(-n)) = 0 \quad i = 1, 2, \quad n > 0$

Then $\mathrm{Pic}(X) \xrightarrow{\sim} \mathrm{Pic}(Y)$

<u>Pf</u>. $\mathrm{Pic}(X) \xrightarrow{a)} \varinjlim \mathrm{Pic}(\mathcal{U}) \xrightarrow{b)} \mathrm{Pic}(\hat{X}_Y) \xrightarrow{c)} \mathrm{Pic}(Y)$

a) A line bdle on $\mathcal{U}$ extends to $X$ by regularity of $X$, and uniquely because $X \setminus \mathcal{U}$ has codim $\geq 1$ $\rightsquigarrow$
$\rightsquigarrow$ it is a closed ~~subset~~ subset of $X$ avoiding the ample divisor $Y$.

b) is an isom. by Thm 1 and 2.

c) ~~are~~ Compare at finite levels

$$0 \to \mathcal{O}(-n)|_{Y_0} \to \mathcal{O}^*_{Y_n} \to \mathcal{O}^*_{Y_{n-1}} \to 0$$

$$f \longmapsto 1 + f^n$$

$\int$ cohom.

$$H^1(Y, \mathcal{O}(-n)) \to H^1(Y_n, \mathcal{O}^*_{Y_n}) \to H^1(Y_{n-1}, \mathcal{O}^*_{Y_{n-1}}) \to H^2(Y, \mathcal{O}(-n))$$

$H^1(Y, \mathcal{O}(-n)) = 0$, $H^2(Y, \mathcal{O}(-n)) = 0$

$$\Downarrow$$

$$\mathrm{Pic}(Y_n) \simeq \mathrm{Pic}(Y_{n-1})$$

$$\mathrm{Pic}(\hat{X}_Y) = \varprojlim_n \mathrm{Pic}(Y_n) \xrightarrow{\sim} \mathrm{Pic}(Y)$$

---

## 1 Foliations in differential geometry

Let $M$ be a $C^\infty$ manifold, $U_\alpha$ covering, $\varphi_\beta \circ \varphi_\alpha^{-1}$ transition maps.

<u>Foliated diffeo</u>: $\mathcal{U}, \mathcal{U}' \subseteq \mathbb{R}^n$, $d \leq n$, $\mathbb{R}^n = \mathbb{R}^d \times \mathbb{R}^{n-d}$

$$\begin{tikzcd} & \mathbb{R}^d \\ \mathbb{R}^n \arrow[ur, "p"] \arrow[dr, "q"'] & \\ & \mathbb{R}^{n-d} \end{tikzcd}, \quad \Psi : \mathcal{U} \to \mathcal{U}'$$

TFAE

- $D\Psi$ takes values in the subspace $M_n(\mathbb{R})$ consisting $\begin{pmatrix} A & B \\ \underline{0} & D \end{pmatrix}$
- $\forall P \in \mathcal{U}$, $\exists$ open $I \subseteq \mathbb{R}^d$ containing $p(P)$
  $J \subseteq \mathbb{R}^{n-d}$ " $q(P)$

  such that $I \times J \subseteq \mathcal{U}$ and $\exists$ $C^\infty$ mappings $\Psi_1 : I \times J \to \mathbb{R}^d$
  $\Psi_2 : J \to \mathbb{R}^{n-d}$

  s.t. $\Psi(x,y) = (\Psi_1(x,y), \Psi_2(y))$
- Given $y' \in \mathbb{R}^{n-d}$, the conn. components of $(q \circ \Psi)^{-1}(y')$ of dim $d$ in $\mathcal{U}'$ are of the form $\underset{\mathbb{R}^d}{\underset{\cap}{\Omega}} \times \{y'\}$

<u>Def</u> $\to$ We say that $\Psi$ is <u>foliated</u> of dim $(d,n)$

Let $M$ be a $C^\infty$ manifold, a <u>foliated atlas</u> of dim $(d,n)$ in $M$ is an atlas for which the transition maps $\varphi_\beta \circ \varphi_\alpha^{-1}$ are foliated of dim $(d,n)$

- A foliation $\mathcal{F}$ on $M$ is an equivalence class of foliated atlases on $M$.

<u>Leaves</u> $(M, \mathcal{F})$ as before, foliated dim $(d,n)$

Look up $M^{\mathcal{F}}$ as follows: equip $M$ with the unique topology finer than the one on $M$ s.t.

given any foliated chart $\varphi : (\mathcal{U} \subseteq M) \to (V \subseteq \mathbb{R}^n)$
$$\mathbb{R}^n = \underbrace{\mathbb{R}^d}_{\text{standard top.}} \times \underbrace{\mathbb{R}^{n-d}}_{\text{discrete top.}}$$

$M^{\mathcal{F}}$ is a $d$-dim. $C^\infty$-manifold

Def. The connected components of $M^{\mathcal{F}}$ are the leaves of $(M, \mathcal{F})$.

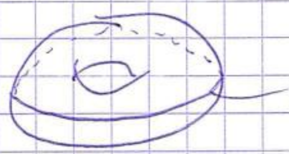
Given a leaf $L$, $L \hookrightarrow M$ is an "injective immersion"

⚠ image may not be closed. Can be dense.

Eg: 1) submersions:

dim $n$ $M$ $\qquad$ $M \longrightarrow N$ submersion.

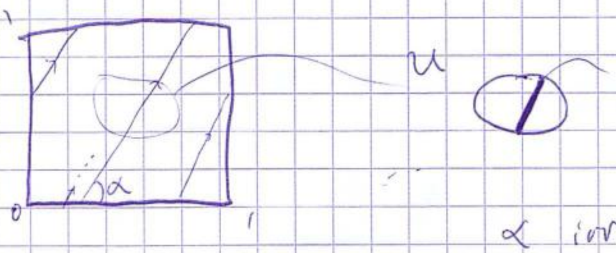
dim $n-d$ $N$ $\qquad$ The connected components of the fibres give a foliation with $d$-dim. leaves.



N $\qquad$ Fibers:

2) $G$ - Lie gp, $H$ closed Lie subgp: $H \to G \to G/_H$

Torus




$\alpha$ rational $\Rightarrow$ for small $\mathcal{U}$, every leaf is connected

$\alpha$ irr $\Rightarrow$ countably many components each leaf.

Frobenius integrability

Def. $M$ $C^\infty$-manifold, tangent bundle $TM$. An involutive distribution of rank $d$ on $M$ is a submodule $F \subseteq TM$ of rank $d$, closed under Lie brackets ie $[F, F] \subseteq F$.

A chart for $M$ adapted to $\mathcal{F}$ is a chart $\mathcal{U} \xrightarrow{\varphi} V$ s.t. $\forall x \in \mathcal{U}$,

$[x, y](f) = x(y(f)) - y(x(f))$

$$D\varphi(x): T_x M \to T_x \mathbb{R}^n : F_x \mapsto \mathbb{R}^d \times \{0\}$$

Thm. With $M, F$ as above, $\forall p \in M$, there is a chart $\varphi$
$\varphi : \mathcal{U} \subset M \to V \subseteq \mathbb{R}^n$ adapted to $F$

Given $(M, \mathcal{F})$, get $F \subset TM$ as follows

$\forall p \in M$, $F_p :=$ tangent sp. to $\mathcal{F}_p$

Conversely, given an involutive resolution $F$ of rank $d$, the atlas of charts adapted to $F$ defines a foliation of dim $(d, n)$ on $M$.

Def. Tangent bdle $T_{\mathcal{F}} :=$ subbundle of $TM$ defined via this correspondence.

Pf F.T. (sketch)

$p \in M$, $\varphi : \mathcal{U}_0 \to \mathbb{R}^n$

$p \mapsto 0$

Assume $\forall i \leq d$, there is a unique regular section $v^i$ of $F|_{\mathcal{U}_0}$ of the form

$$v^i = \frac{\partial}{\partial x_i} + \sum_{k=d+1}^{n} a^i_k \frac{\partial}{\partial x_k}, \quad a^i_k \in C^\infty(\mathcal{U}_0, \mathbb{R}).$$

$[v^i, v^j] = 0$ ($\Leftarrow$ because of involution)

This gives us commuting vector fields

Algebraic setting

Let $k$ field, $X/_k$ sm quasi-proj. variety, $F$ subvector bundle of $T_{X/k}$.

Def. If $F$ is stable under Lie bracket, $(X, F)$ is an algebraic foliation.

Def. If $F$ is a saturated coh. subsheaf of $T_{X/k}$ stable under Lie bracket, $(X, F)$ is an algebraic foliation with singularities.

Rem. $\exists\, \mathcal{U} \subset X$, $X \setminus \mathcal{U}$ codim $\geq 2$ s.t. $(\mathcal{U}, F|_{\mathcal{U}})$ is an algebraic foliation.

Conversely, given $(X, F)$ — , $X \hookrightarrow \bar{X}$ smooth compactific.
$(\bar{X}, \bar{F})$ — with singularities.

If $k = \mathbb{C}$, $(X, F)$ algebraic foliation $\Rightarrow$ $(X(\mathbb{C}), F^{an})$ complex analytic foliation

Similar if char $k = 0$

Let $(X, F)$, $p \in X(k)$

Formal germ of leaf through ~~p~~ p.

$x_1, \dots, x_n$ on $\mathcal{U} \ni p$ induce an étale morphism to $\mathbb{A}^n_k$

One defines flow with exponencials.

⋮

Prop/Def: TFAE

1) The formal leaf $\hat{\mathcal{F}}_p$ of $(X, F)$ through $P$ is algebraic.

2) $\exists\, V \hookrightarrow X$ smooth integral ~~subsch~~ subscheme of dim $d$ invariant under $F$. Here $\hat{F}_p$ is said the algebraic leaf of $F$ through $p$.

---

Some geometric applications of algebraization thms

(I) Algebraicity of certain foliations in char. 0

Idea: - use a relative version of Frobenius integrability / formal exponential to construct formal schemes along a subscheme

- Use Hartshorne's thm.

$\leadsto$ Lie subalg. of smooth alg. groups over function fields.

- Let $k$ field char 0, $C$ smooth proj. connected over $k$.

$(\mathcal{G}, \varepsilon)$ sm quasi-proj. group scheme over $C$.

Eg: $GL_n/\mathbf{C}$, $V/C$ vector bundle, $Aut(V)/C$, abelian schemes, semi-abelian sheaves, Neron models.

Let $(G, e) := \mathcal{G}_K$ generic fiber, which is a sm alg gp ($K = k(C)$)

$\text{Lie}\,\mathcal{G}$ vb over $C$.

$$\begin{array}{ccc} \{\text{sub v.b. of Lie}\,\mathcal{G}\} & \xrightarrow{\sim} & \{\text{sub v.s. of Lie}\,G\} \\ \cup & & \cup \\ \{\text{sub Lie algebras of Lie}\,\mathcal{G}\} & \xrightarrow{\sim} & \{\text{sub Lie alg of Lie}\,G\} \end{array}$$

Thm. (Bost) Let $\mathbf{F}$ be a sub Lie algebra of $G$, $\mathcal{F}$ Lie subgp of $\mathcal{G}$. Assume $\mathcal{F}$ is ample. Then $F$ is an alg. sub Lie algebra of Lie $G$ with associated smooth connected subgroup $H$. Moreover, $H$ is unipotent.

Rem. False if char $k \neq 0$

Cor - If $G$ is an abelian variety, then $G$ has no non-trivial unipotent subgp, Lie $\mathcal{G}$ has no ample subbundles.

- There exists a relative formal exponential map

$$\begin{array}{ccc} \mathbb{V}(\text{Lie}\,\mathcal{G}^\vee)^\wedge_{\mathcal{E}} & \xrightarrow{\sim} & \hat{\mathcal{G}}_\varepsilon \\ \| & & \\ \hat{\mathcal{L}} & & \widehat{\exp}_{\mathcal{G}} \end{array}$$

which is an isomorphism of formal schemes (and if gp is commutative, isom. of formal gp schemes)

~~Not~~ We have

$$\begin{array}{cccc} d\,\widehat{\exp}_{\mathcal{G}} : & T_0\,\hat{\mathcal{L}} & \longrightarrow & T_\varepsilon\,\hat{\mathcal{G}}_e \\ & \wr\| & & \wr\| \\ & \text{Lie}\,\mathcal{G} & = & \text{Lie}\,\mathcal{G} \end{array}$$

$\widehat{\exp}_G : \hat{L} \xrightarrow{\sim} \hat{G}_e$

Let $\hat{\mathbb{V}} := \widehat{\exp}_{\mathcal{G}}(\mathcal{F})$ , $\hat{V} := \widehat{\exp}_{G}(F)$

$\varepsilon^* N_{\varepsilon} \hat{\mathbb{V}} \simeq \mathcal{F}$ ample vector bundle on $C$.

By Hartshorne's thm, $\hat{V}$ is algebraizable, ie its Zariski closure in $\mathcal{G}$ is of dim $=$ dim $F$.

$\mathcal{F}$ abelian $\Leftrightarrow$ $F$ abelian $\xrightarrow{\text{in this case}}$ $\overline{\hat{\mathbb{V}}}^{\text{Zar}}$ is commutative subgp scheme $\mathcal{H}$ of $\mathcal{G}$.

$\mathcal{F} \hookrightarrow \mathrm{Lie}\, \mathcal{G}$

The graph $\Gamma$ of this embedding gives a sub-Lie algebra of $\mathrm{Lie}(\mathbb{V}(\mathcal{F}^\vee) \times \mathcal{G})$. As a v.b. on $C$, it is $\simeq$ to $\mathcal{F}$ ample

Then, $\widehat{\exp}_{-}(\Gamma)$ is algebraizable $\Rightarrow$

$\Rightarrow$ $\tilde{\mathcal{H}} \subset \mathbb{V}(\mathcal{F}^\vee) \times \mathcal{G}$

At the generic fiber,

$$\begin{tikzcd} \tilde{H} \arrow[r] \arrow[d, "\text{fin. surj. étale morphism}"'] & \mathcal{G} \\ \mathbb{V}(F^\vee) \arrow[ur, dashed, bend right, "\text{iso onto its image}"'] & \end{tikzcd}$$

isom.

Hence, $\mathbb{V}(F^\vee) \simeq H$

In general, write Harder-Narashiman of $\mathrm{Lie}\, \mathcal{G}$:

$\{0\} \subseteq E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n = \mathrm{Lie}\, \mathcal{G}$

$\mu_1 > \cdots > \mu_n$ slopes of ~~stable~~ semistable quotients $E_i / E_{i-1}$

Let $E_+ \subset E$ the maximal $E_i$ with $\mu_i \geq 0$. (hence $\mu_j \geq 0$ for $j \leq i$).

Then any ample subbundle of $\mathrm{Lie}\, \mathcal{G}$ is contained in $E_+$. Any subgp of a unipotent alg. gp is unipotent.

We show that $(E_r)_K$ is the Lie alg of a unipotent subgp of Lie G.

<u>Lemma</u> · $\mu_i > 0 \Rightarrow E_{i,K}$ is a sub Lie algebra of Lie G

· $\forall j \leq i$, $E_{j,K}$ is a Lie ideal in $E_{i,K}$ $\left([E_{i,K}, E_{j,K}] \subset E_{j,K}\right)$

· $\mu_i > 0$, $E_{i,K}/E_{i-1,K}$ is abelian

<u>Pf</u>: uses that $\mu_{min}(E) > \mu_{max}(F) \Rightarrow Hom(E,F) = 0$

By the previous case, $E_{i,K}/E_{i-1,K}$ is algebraic.

$\overline{H_{i,K}}$ vector gp with $Lie(\overline{H_{i,K}})$

· $\widehat{exp_G}$ is the formal flow for the foliation on G associated to F.

<u>(II) Kodaira vanishing for $H^1$</u>

<u>Thm</u>. X sm proj. var. of dim $d \geq 0$ over $k$, char 0.

L ample line bundle on X.

Then $H^1(X, L^\vee) = 0$

<u>Rem</u>. General form of Kodaira vanishing: $H^i(X, L^\vee) = 0$, $i < d$

equivalently (Serre duality), $H^j(X, L \otimes \omega_X) = 0$ $j > 0$. (2)

- Kodaira ('53) proved (2) by analytic techniques.
- Deligne - Illusie proved that algebraically.

Meanwhile, algebraic proofs for $H^1$: Mumford ('67) using connectedness for ample divisors and reducedness of Picard variety in char. 0.

Mumford ('77): for $H^1$, using results of Bogomolov on rank 2 v.b.

Pf. Reduction to the case of surfaces.
(Bost)

- For $N \gg 0$, $X$ general member of $|L^{\otimes N}|$
- $Y$ smooth (Bertini)
- $H^1(X, L^{\otimes -N+1}) = 0$ Enriques - Severi - Zariski lemma.

$H^1(X, L^\vee) \hookrightarrow H^1(Y, L^\vee) \Rightarrow$ can assume $\dim X = 2$.

Let $\alpha \in H^1(X, L^\vee)$, represented by $\alpha_{ij} \in Z^1(\mathcal{U}, L^\vee)$ for

$\mathcal{U} = (U_i)_{i \in I}$ open covering. $\alpha_{ij} \in \Gamma(U_{ij}, L^\vee)$

$$\cap$$

$$\mathrm{Hom}(U_{ij}, \mathbb{V}(L^\vee))$$

$V := \mathbb{V}(L^\vee)$,

$\hat{V} := \hat{V}_0$ $\exp(\alpha_{ij}) := \sum_{n \geq 0} \frac{1}{n!} \alpha_{ij}^n \in \Gamma(U_{ij}, \mathcal{O}_{\hat{V}})$

$\alpha_{ij} \in Z^1(\mathcal{U}, L^\vee) \Rightarrow (\exp(\alpha_{ij})) \in Z^1(\mathcal{U}, \mathcal{O}^\times_{\hat{V}})$

$\Rightarrow \exp(\alpha) \in H^1(\hat{V}, \mathcal{O}^\times_{\hat{V}})$

$\rightsquigarrow$ $\hat{E}$ formal line bundle on $\hat{V}$

In fact, $\exp(\alpha_{ij}) \in \Gamma(U_{ij}, \ker(\mathcal{O}^\times_{\hat{V}} \longrightarrow \mathcal{O}^\times_X))$

$\rightsquigarrow \hat{E}|_X = 0$

- $X$ smooth $\Rightarrow \varepsilon^* : \mathrm{Pic}(V) \xrightarrow{\sim} \mathrm{Pic}(X)$

Assume that $\hat{E}$ is alg., i.e. $\exists$ $E$ line bdle on $V$ with $E|_{\hat{V}} \simeq \hat{E}$

Then, $\hat{E} \simeq 0$. Hence $\exp(\alpha) = 0 \Rightarrow \alpha = 0$.

$\hookrightarrow$ by restricting to $V$, Algebraization of $\hat{E}$.

- $\dim(X) \geq 2 \Rightarrow \dim(V) \geq 3$
- We embed $V$ into a proj. cone $V^*$ over $X$, s.t. $N_{V^*}X$ is ample (Grauert)
- Then apply the existence thm $\Rightarrow \hat{E}$ is algebraizable.

(III) Lefschetz thm for F-divided sheaves

- Let $k$ be a field of char $p > 0$

Def. $X/_k$ sm q-proj. variety. An F-divided sheaf $E.$ on $X$ is a collection of coherent sheaves $E_n$ together with isomorphisms $F^* E_{i+1} \xrightarrow{\sim} E_i$ with $F$ absolute Frob. of $X$.

↳ This gives a good notion ~~of~~ on positive char of local systems

Thm. $X$ smooth

(i) $E. \in F\text{-Div}(X) \Rightarrow \forall n \geq 0$, $E_n$ loc. free

(ii) $F\,\text{Div}(X)$, equipped with the monoidal structure $(E. \otimes F.)_n := E_n \otimes F_n$ is an ab. Tannakian category

(iii) (Katz) $F\text{-Div}(X)$ is equiv. to the category of integrable connections

↳ The fiber functor comes from $x \in X(k) \Rightarrow x^* : F\text{Div}(X) \to \mathrm{Vec}_k$,

$\pi^{alg}(X, x) := \pi(F\text{Div}(X), x^*)$

Question (Gieseker): If $\pi_1^{\acute{e}t}(X) = 0$, does $\pi_1^{alg}(X) = 0$?

In char. 0, $\pi_1^{alg}(X)$ is $\pi_1^{\acute{e}t}(X)$

Answer (~~Bost,~~ Esnault, ~~Srinivas~~ Mehta, Hrushovski) ⇝ Yes if $X$ sm. proj

Thm (Bost, Esnault, Srinivas): $X$ smooth proj. of pure dim. $d$, $Y$ smooth ample divisor, $x \in Y(k)$. Then $\iota_*: \pi_1^{alg}(Y, x) \to \pi_1^{alg}(X, x)$ is an isom. if $\dim \geq 3$, surjective if $\dim = 2$.

For $d=2$, Tannakian formalism:

$\iota_*$ surjective $\Leftrightarrow$ · $\iota^* : FDiv(X) \to FDiv(Y)$ fully faithful

· For $E_\bullet \in FDiv(X)$, $V'_\bullet \subset \iota^* E_\bullet$ sub F-div. sheaf, then $\exists\, V_\bullet \subset E_\bullet$ with $\iota^* V = V'$.

Idea:

$$FDiv(X) \to \operatorname{colim}_{Y \subset U \subset X} FDiv(U) \to FDiv(\hat{X}_Y) \to FDiv(Y)$$

extension algebraization deformation

This one is an equiv. of categories

<u>Thm</u> With above assumptions, $E$ coh. sheaf on $X$, $\hat{E} := E|_{\hat{X}_Y}$ v.b.
Then any subquot. bundle of $\hat{E}$ is algebraizable.

=

## Arakelov geometry

(I) Hermitian vector bundles over $\operatorname{Spec} \mathcal{O}_K$

(II) Arakelov degree and slope

(III) Height and slope inequalities.

Let $K$ # field, $\Sigma_K :=$ places of $K$ $= \Sigma_{K,\infty} \sqcup \Sigma_{K,f}$

$\Sigma_{K,\infty}$: embeddings $\sigma : K \hookrightarrow \mathbb{C}$, $|\cdot|_\sigma$ abs. value; $\Sigma_{K,f}$: $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$

<u>Recall</u>: product formula: $x \neq 0 \leadsto \prod_{v \in \text{places}} |x|_v = 1$

Def. A hermitian vb / $\mathrm{Spec}\, \mathcal{O}_K$ is
$$\overline{E} = \left(E, \{\|\cdot\|_\sigma\}_{\sigma \in \Sigma_{K,\infty}}\right)$$

Projective $\mathcal{O}_K$-module of fin. rank

for any $\sigma : K \hookrightarrow \mathbb{C}$, $\|\cdot\|_\sigma$ is a hermitian norm on $E_\sigma := E \otimes_{\mathcal{O}_K, \sigma} \mathbb{C}$ invariant by complex conjugation.

ie $\|\bar{e}\|_{\bar{\sigma}} = \|e\|_\sigma$

Rem 1) This data ~~amounts~~ gives rise to the data of $E$ and of a hermitian norm invariant by complex conjugation on ~~$E_{\mathbb{C}}$~~ $E_{\mathbb{C}} = E \otimes_{\mathbb{Z}} \mathbb{C}$

2) For $\overline{E}$ hermitian vb over $\mathcal{O}_K$, and $\mathfrak{p} \in \mathcal{O}_K$ prime, one can define a $\mathfrak{p}$-adic norm of $E_\mathfrak{p} := \underbrace{K_\mathfrak{p}}_{\text{(completion of } K \text{ wrt } |\cdot|_\mathfrak{p})} \otimes_{\mathcal{O}_K} E$

via $\|x\|_\mathfrak{p} = \inf \{ |a|_\mathfrak{p},\ a \in K \mid ax \in \mathcal{O}_{K_\mathfrak{p}} \otimes_{\mathcal{O}_K} E \}$

Classical constructions

- Pullback: if $f : \mathrm{Spec}\, \mathcal{O}_L \to \mathrm{Spec}\, \mathcal{O}_K$, $\overline{E}$ hermitian vb / $\mathcal{O}_K$, then $f^*\overline{E}$ is a hermitian vb over $\mathcal{O}_L$.
- Direct sum: $\overline{E}_1, \overline{E}_2$ herm. vb $\rightsquigarrow$ $\overline{E}_1 \oplus \overline{E}_2$ hermitian vb
- Subbundles: restrict the ambient metrics
- Quotient: ⚠ we have to start from saturated $\mathcal{O}_K$-modules: $F = F_K \cap E$
- Tensor product: $T^k \overline{E}$: $k^{th}$ tensor power of $E$
- Symmetric powers: $S^k \overline{E}$
- Exterior " : $\Lambda^k \overline{E}$, with $\underset{(a_1 \wedge \cdots \wedge a_k)}{\text{norm}} = \det(a_i \cdot a_j)$
- Duals: $\overline{E}^\vee$

(II)

**Def.** Let $\overline{E}$ be an h. v.b. over $\mathcal{O}_K$.

- Assume that $\mathrm{rk}\, E = 1$. The <u>Arakelov degree</u> of $\overline{E}$ is

$$\widehat{\deg}\, \overline{E} := \log\left(\# E/\mathcal{O}_K e\right) - \sum_{\sigma \in \Sigma_{K,\infty}} \log \|e\|_\sigma$$

where $e \in E \setminus \{0\}$, $\log \|e\|_\sigma = \varepsilon_\sigma \log|\sigma(e)|$, $\varepsilon_\sigma = \begin{cases} 1 & \text{if } \sigma(K) \subset \mathbb{R} \\ 2 & \text{else} \end{cases}$

<u>Rem</u> 1) This doesn't depend on the choice of $e$

2) $\widehat{\deg}\, \overline{E} = -\sum_{v \in \Sigma_K} \log \|e\|_v$

3) Normalized degree: $\widehat{\deg}_n \overline{E} := \frac{1}{[K:\mathbb{Q}]} \widehat{\deg}(\overline{E})$

If $\overline{E}$ is of rank $r \geq 1$, then

$$\widehat{\deg}(\overline{E}) := \widehat{\deg}\left(\Lambda^r \overline{E}\right)$$

<u>Properties of $\widehat{\deg}$</u>

Let $\overline{L_1}, \overline{L_2}$ herm. line bdles $/ \mathcal{O}_K$.

$$\widehat{\deg}(\overline{L_1} \otimes \overline{L_2}) = \widehat{\deg}(\overline{L_1}) + \widehat{\deg}(\overline{L_2})$$

$$\widehat{\deg}(\overline{L_1}^\vee) = -\widehat{\deg}(\overline{L_1})$$

<u>Lem.</u> Let $\overline{E_1} \subseteq \overline{E_2}$ h. vb over $\mathcal{O}_K$, $\overline{E_1}$ saturated. Then

$$\widehat{\deg}(\overline{E_2}) = \widehat{\deg}(\overline{E_1}) + \widehat{\deg}\left(\overline{E_2}/\overline{E_1}\right)$$

<u>Def.</u> The slope of $\overline{E}$ is

$$\hat{\mu}(\overline{E}) := \widehat{\deg}(\overline{E}) / \mathrm{rk}\, E$$

<u>Property</u>: $\hat{\mu}(\overline{E_1} \otimes \overline{E_2}) = \hat{\mu}(\overline{E_1}) + \hat{\mu}(\overline{E_2})$

**Def.** The maximal slope of $\overline{E}$ is
$$\widehat{\mu}_{max}(\overline{E}) := \sup\{\widehat{\mu}(\overline{F}),\ \overline{F} \text{ non-zero subbdle of } \overline{E}\}$$

The minimal slope of $\overline{E}$ is
$$\widehat{\mu}_{min}(\overline{E}) := \inf\{\widehat{\mu}(\overline{E}/\overline{F}),\ \overline{F} \text{ saturated subbdle of } \overline{E}\}$$

**Rem.** $\widehat{\mu}_{max}(\overline{E}^{\vee}) = -\widehat{\mu}_{min}(\overline{E})$

**Lem.** Assume $(e_1,\dots,e_r)$ is a basis of $E_K$ in $E$. Then
$$\widehat{\mu}_{min}(\overline{E}) \geq -\sum_{\sigma\in\Sigma_{K,\infty}} \log\left(\max_{1\leq i\leq r}\|e_i\|_\sigma\right)$$

Pf. def + Hadamard inequality.

**Lem** Assume we have a decreasing filtration of $E = E^0 \supset \dots \supset E^{n+1} = 0$ by saturated $\mathcal{O}_K$-modules.
Set $\overline{F_i} := \overline{E_i/E_{i+1}}$. Then
$$\widehat{\deg}\,\overline{E} = \sum_{i=0}^{n}\widehat{\deg}(\overline{F_i})$$

**Lem.** Let $\overline{E}$ be a h.v.b. over $\mathcal{O}_K$. There is a real number $c\in\mathbb{R}$ s.t. $\forall k\in\mathbb{N}^*$,
$$\widehat{\mu}_{max}(S^k\overline{E}) \leq c\,k$$

Exercises: Show that
$$c = \sum_{\sigma\in K,\infty}\log\left((\mathrm{rk}\,E)\cdot\max_{1\leq i\leq r}\|e_i\|_\sigma\right)$$

**Rem.** Bost has a better bound:
$$\widehat{\mu}_{max}(S^k\overline{E}) \leq k\left[\widehat{\mu}_{max}(\overline{E}) + 2\,\mathrm{rk}\,\overline{E}\cdot\log(\mathrm{rk}\,\overline{E})\right]$$

## (2) The Harder-Narasimhan-Grayson-Stuhler (or canonical) filtration.

Let $V$ be the closed convex envelop in $\mathbb{R}^2$ of
$\{(\mathrm{rk}\,\bar F, \widehat{\deg}\,\bar F),\ \bar F \text{ subbdle of } \bar E\}$

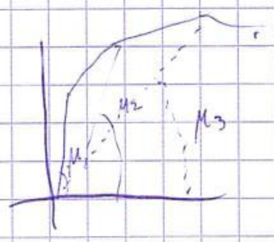
There is a concave function $P_{\bar E} : [0, \mathrm{rk}\,\bar E] \to \mathbb{R}$ s.t.

- $P_{\bar E}$ is affine on each $[i, i+1]$ for $i = 0, \dots, \mathrm{rk}\,\bar E - 1$.
- $P_{\bar E}(0) = 0$, $P_{\bar E}(\mathrm{rk}\,\bar E) = \widehat{\deg}\,\bar E$
- $V$ coincides with
$$\{(x,y) \in [0, \mathrm{rk}\,E] \times \mathbb{R} \mid y \leq P_{\bar E}(x)\}$$

Def. The canonical ~~filtration~~ polygon of $\bar E$ is the graph of $P_{\bar E}$.

Thm (Stuhler, Grayson) Let $i_1 < \dots < i_k$ be the jumps of $P_{\bar E}$.
For any $j \in [1, k]$, $\exists !$ $O_K$-submodule $E_j$ in $E$ s.t.
$$\mathrm{rk}\,E_j = i_j \quad \text{and} \quad P_{\bar E}(i_j) = \widehat{\deg}(\bar E_j)$$

We have in this way a filtration of $\bar E$
$$\{0\} = E_0 \subset E_1 \subset \dots \subset E_{k-1} \subset E_k = E$$
s.t. the quotient $\bar E / E_i$ are torsion free.




Rem. If we set $\hat\mu_i = \hat\mu(\bar E_i / \bar E_{i-1})$, then $\hat\mu_1 > \dots > \hat\mu_k$

Def. $\bar E$ semistable iff the canonical polygon is a line.

## III

Let $\bar E, \bar F$ h.v.b over $O_K$, $\varphi : E_K \to F_K$ $K$-linear non-zero map

Def. The height of $\varphi$ is
$$h(\varphi) := \sum_{v \in \Sigma_K} \log \|\varphi\|_v$$

Lem. Standard slope inequality $\varphi : E_K \to F_K$

Assume that $\varphi$ is inj. Then

$$\hat{\mu}(\bar{E}) \le \mu_{max}(\bar{F}) + h(\varphi)$$

We also have a filtered version

Relation to Siegel's lemma:

Lem (Siegel): Let $A \in M_{n,r}(\mathbb{Z})$, $n > r$
columns

Then there is $x \in \mathbb{Z}^n \setminus \{0\}$ s.t.
$$\begin{cases} Ax = 0 \\ \max |x_i| \le (n\, a)^{r/n-r}, \text{ where} \\ a = \max |a_{ij}| \end{cases}$$

Let $E = \mathbb{Z}^n$ be the standard lattice endowed with the usual quadratic form $q_E : x \longmapsto \sum^n x_i^2$

Let $\bar{F} = \mathbb{Z}^r$ similar.

Want to Apply slope inequality to $\alpha_{\mathbb{Q}} : E_K \to F_K$ defined by $A$, but $\alpha$ not injective.

Let $E_1 = \ker \alpha \cap E$, $\bar{\alpha} : \left(E/E_1\right)_K \to F_K$. Then we have

$$\hat{\mu}\left(\overline{E/E_1}\right) \le \hat{\mu}(\bar{F}) + h(\bar{\alpha})$$

Computations $\rightsquigarrow$

$$\cdot\ \hat{\mu}(\bar{E}_1) \geq \frac{-r}{n-r} \log\left(\sqrt{r} \cdot a\right)$$

Minkowski thm $\to$ $\hat{\mu}(\bar{E}_1) \geq \log t + c \implies \exists x \in E_1 \setminus \{0\} : q_E(x)^{1/2} \le t^2$

$$\rightsquigarrow \exists x \in \mathbb{Z}^n \setminus \{0\}. \quad Ax = 0$$
$$\|x\| \le \left(a\sqrt{r}\right)^{\frac{r}{n-r}}$$

Rem. $K = \mathbb{Q}$,

$\overline{E} = (E, \|\ \|)$ $\longrightarrow$ euclidean norm in $\mathbb{R}^{\mathrm{rk} E}$.

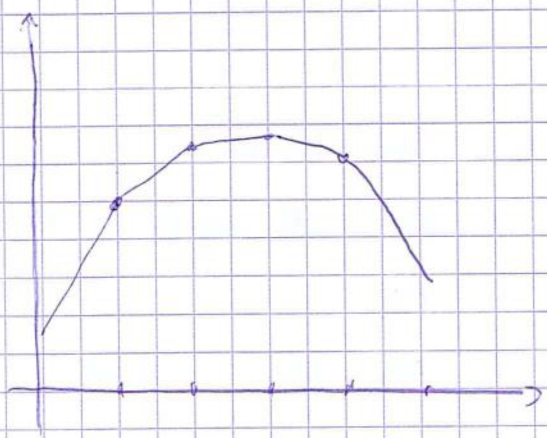
Consider covol $\overline{E}$, then

$$\widehat{\deg}\, \overline{E} = -\log(\mathrm{covol}\, \overline{E})$$

Let $\lambda_1(\overline{E}) := \min_{v \in E \setminus \{0\}} \|v\|$ (large $\lambda_1 \hookrightarrow$ negative v.b in alg. geom).

$$\mu_{\min}(\overline{E}) = -\log \lambda_1(\overline{E}) + \underbrace{(\text{error term})}_{\leq C(\mathrm{rk}\, E)}$$

Diego's talk. Rem: Formalism slope appears in many settings



We need, for slope formalism, just

- Additive $\mathrm{rk} : \mathcal{L} \to \mathbb{N}$
- Degree function $\deg : \mathcal{L} \to \mathbb{R}$ satisfying $\deg(E_1 + E_2) + \deg E_1 \wedge E_2 \geq \deg E_1 + \deg E_2$

---

Arakelov in surfaces:

- Hironaka's thm on formal functions

- Thm (Hironaka '68) $k = \bar{k}$ field, $X$ sm. proj. connected surface$/k$, $Y \hookrightarrow X$ sm. proj. conn. curve. $\hat{X}_Y =: \hat{X}$ completion

If $Y\cdot Y>0$, then $K(X)\hookrightarrow K(\hat{X})$ is trivial

- Let $K :=$ sheaf on $|\hat{X}|=Y$ which associates to $\mathcal{U}\subseteq Y$ open affine, the fraction field of $\mathcal{O}_{\hat{X}}(\mathcal{U}) = \varprojlim_n \mathcal{O}_{Y_n}(\mathcal{U})$

Any element in $K(X)$ induces an element of $K(\hat{X})$

~~Pf~~

Eg: $X = Bl_p(\mathbb{P}^2)$, $Y=$ exc. divisor

$$K(X) = K(\mathbb{P}^2) \neq K(\hat{X}) \quad !$$

Pf. 2 steps:

(1) $K(X)\hookrightarrow K(\hat{X})$ is algebraic — not today

(2) $K(X)\hookrightarrow K(\hat{X})$ is "algebraically closed"

Rem: Function field application

Let $B =$ smooth proj. conn. curve$/k$, "base curve"

$\pi: \mathfrak{X} \longrightarrow B$ "fibered surface", $\pi$ $\begin{cases} \text{- flat, } \mathfrak{X} \text{ regular 2-dim scheme} \\ \text{- proj.} \\ \text{- geom. fibres connected} \end{cases}$

Let $P\in \mathfrak{X}(B)$ a section.

If $f\in \widehat{\mathcal{O}_{X,P}}$ extends to a formal rational function on $\hat{\mathfrak{X}}_P$

$\deg(P^* T_\pi)>0$, then $f$ is a formal germ of a rat'l function on $X$.

Pf. $\deg(P^* T_\pi) = P\cdot P>0$ $\square$

Pf (Step (2) of Thm)

Lemma (Connectedness) Let $X$ be a sm. proj connected surface$/k$, $D\geq 0$ divisor. If $D$ is big and nef, then $\mathrm{Supp}(D)$ is connected.

$D.D>0$, $\forall\, C\hookrightarrow X$ integral curve, $D.C\geq 0$

Pf. 1) $D$ is num connected.

Write $D = D_1 + D_2$, $D_i \neq 0$, $D_i \geq 0$

To show: $D_1 \cdot D_2 > 0$.

Assume $D_1 \cdot D_2 \leq 0$. $D$ nef $\Rightarrow$ $D \cdot D_1 \geq 0$ $\Rightarrow$ $D_1 D_1 \geq - D_1 \cdot D_2$

$D \cdot D_2 \geq 0$ $\qquad D_2 \cdot D_2 \geq - D_1 \cdot D_2$

$$\Rightarrow (D_1 \cdot D_1)(D_2 \cdot D_2) \geq (D_1 \cdot D_2)^2 \Rightarrow \det \begin{pmatrix} D_1 \cdot D_1 & D_1 \cdot D_2 \\ D_1 \cdot D_2 & D_2 \cdot D_2 \end{pmatrix} \geq 0$$

HIT v3.0 $\rightsquigarrow$ since $D^2 > 0$,

$$\det \begin{pmatrix} & \end{pmatrix} \leq 0$$

Hence $[D_1] = \lambda [D_2]$ for some $\lambda \in \mathbb{R}_{>0}$

$\hookrightarrow$ in $\mathrm{Num}(X)_{\mathbb{R}}$

Now $\lambda D_1 \cdot D_1 = D_1 D_2 \leq 0 \Rightarrow (1+\lambda)^2 D_1 \cdot D_1 \leq 0$

$\qquad \qquad \qquad \qquad \qquad \qquad \|$

$(1+\lambda) D_1 \cdot (1+\lambda) D_1 = (D_1 + D_2)^2 = D^2 > 0$ ↯

Lemma-1

Pf ②: Let $\varphi \in K(\hat{X})$ which is alg. over $K(X)$.

$K(X) \subseteq (K(X))(\varphi)$ is finite. $X' =$ normalization of $X$ in $K(X)(\varphi)$

$$\begin{array}{c} X' \\ \downarrow f \text{ finite} \\ X \hookrightarrow Y \end{array}$$

Lem $\Rightarrow$ $f$ ~~birational~~ $f^{-1}(Y)$ connected

$\rightsquigarrow$ $f$ birational

Hence, $K(X)(\varphi) = K(X) \Rightarrow \varphi \in K(X)$ $\square$

Aim: State and ~~pf~~ prove the arithmetic analogue of step 2.

- Intersection theory
- HIT

$\Big\}$ Arakelov

# Fibered surfaces

B is base curve: $\mathbb{A}^1_k$, $\mathbb{P}^1_k$, $\mathrm{Spec}\begin{cases}\mathbb{Z}_p \\ \mathbb{Z} \\ \mathbb{C}[[t]]\end{cases}$ etc

integral noeth'n regular one dim. scheme

$\mathcal{X} \xrightarrow{\pi} B$ flat proj. (surjective), $\mathcal{X}$ integral regular scheme of dim. 2 with geom. conn. fibers.

Eg: $\begin{array}{c}\mathbb{P}^1_B \\ \downarrow \\ B\end{array}$ $\quad \begin{array}{c}\mathcal{X} = Z_+\left(y^2 z = (x-z)(x-\lambda z)\right) \\ \downarrow \\ \mathbb{A}^1_{\mathbb{C}} \ni \lambda\end{array}$

**Def.** $\begin{array}{c}\mathcal{X} \\ \downarrow \\ B\end{array}$

$D \hookrightarrow X$ integral domain

$$\pi(D) = \begin{cases}\{pt\} & D \text{ is vertical} \\ B & D \text{ is horizontal}\end{cases}$$

Let $b \in B$ closed pt.

**Def.** $\mathrm{Div}_b(\mathcal{X}) = \{D \in \mathrm{Div}(\mathcal{X}) \mid \pi(D) = \{b\}\} \subseteq \mathrm{Div}(\mathcal{X})$ subgp.

$\exists\ \mathrm{Div}(\mathcal{X}) \times \mathrm{Div}_b(\mathcal{X}) \xrightarrow[\text{pairing}]{\text{inters.}} \mathbb{Z}$

**Thm** (Liu, 9.1.12) $\exists!$ bilinear map $\mathrm{Div}(\mathcal{X}) \times \mathrm{Div}_b(\mathcal{X}) \xrightarrow{i_b} \mathbb{Z}$ s.t.
(book)

1) $D \in \mathrm{Div}(\mathcal{X})$, $E \in \mathrm{Div}_b(\mathcal{X})$ have a common component

$$i_b(D,E) = \sum_{x \in \mathcal{X}} \underbrace{i_x(D,E)}_{\mathrm{length}_{\mathcal{O}_{\mathcal{X},x}}\left(\mathcal{O}_{\mathcal{X},x}/(I_{D,x} + I_{E,x})\right)} [k(x):k(b)]$$

2) $i_b|_{\mathrm{Div}_b(\mathcal{X}) \times \mathrm{Div}_b(\mathcal{X})}$ is symmetric

3) If $D, D' \in \mathrm{Div}(\mathcal{X})$ s.t. $D = D' + \mathrm{div}(f)$, $f \in K(\mathcal{X})$,

then $i_b(D, E) = i_b(D', E)$

4) If $0 \leq E \leq \mathcal{X}_b$, then $i_b(D, E) = \deg \mathcal{O}_{\mathcal{X}}(D)|_E$

5) $\langle \cdot, \cdot \rangle$ $\mathrm{Div}_b(\mathcal{X})_{\mathbb{R}} \times \mathrm{Div}_b(\mathcal{X})_{\mathbb{R}} \to \mathbb{R}$ is neg. semi-def.

Message: we can do intersection theory in fibered surfaces!

<u>Ex</u>. Assume $\mathrm{Pic}(B) = 0$

$b \in B$ is a ppal divisor.

Then $\mathcal{X}_b = \pi^* b$ is a ppal divisor. If $E$ is vertical,

$3) \Rightarrow \pi^* b \cdot E = 0$. $E =$

Rem. If $P : B \to \mathcal{X}$ is a section, then $\mathcal{X}_b \cdot P = 1 \neq 0$

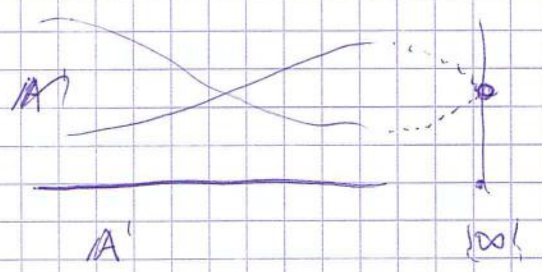
Hence linear equiv. is not respected $\rightsquigarrow$ not the right notion!

Eg: $B = \mathbb{A}^1_{\mathbb{C}}$,

$$\begin{array}{ccc} \mathcal{X} & \hookrightarrow & \overline{\mathcal{X}} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \overline{\pi}} \\ b \in \mathbb{A}^1_{\mathbb{C}} & \hookrightarrow & \mathbb{P}^1_{\mathbb{C}} \end{array}$$

intersection pairing extend to $\mathrm{Div}(\overline{\mathcal{X}}) \times \mathrm{Div}(\overline{\mathcal{X}}) \to \mathbb{Z}$

$\ni b$ is not ppal.




$$\begin{array}{ccc} \mathcal{X} & \hookrightarrow & \widehat{\mathcal{X}} \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, \mathbb{Z} & \overset{?}{\hookrightarrow} & \mathrm{Spec}\, \widehat{\mathbb{Z}} \end{array}$$

?

What do we do? Complex analysis on $\mathcal{X}(\mathbb{C})$ Riemann surf

Arakelov's intersection pairing

Analytic part: $X$ compact Riemann surf., $g(X) \geq 1$

Let $\omega_1, \ldots, \omega_g$ basis of $H^0(X, \Omega^1)$ orthonormal

$$\langle \omega, \eta \rangle = \frac{i}{2} \int_X \omega \wedge \bar{\eta}$$

Def. $\mu_{Ar} := \frac{i}{2g} \sum_{k=1}^{g} \omega_k \wedge \overline{\omega_k}$ independent of choice of basis.

$$\int \mu_{Ar} = 1$$

Thm. Def $\exists!$ $C^\infty$-function $gr_X : (X \times X) \setminus \Delta \to \mathbb{R}$ s.t.

i) $\forall y$, $\partial \bar{\partial} gr_X(x, y) = 2\pi i \, \mu_{Ar}(x)$ $\forall x \neq y$ in $X$.

ii) $\int_X gr_X(x, y) \, \mu_{Ar}(x) = 1$

Green functions

Rem $gr_X$ are symmetric (use Green's formula).

Arakelov's intersection ~~the~~ pairing ($B = \operatorname{Spec} \mathcal{O}_K$)

$\pi : \mathcal{X} \to B = \operatorname{Spec} \mathbb{Z}$

$\widehat{\mathrm{Div}}(\mathcal{X}) = \mathrm{Div}(\mathcal{X}) \oplus \mathbb{R} \cdot F_\infty$ (fiber at $\infty$ geometrically)

here, formal sym. b. $1$ which minimizes the fiber at $\infty$

$\hat{D} \in \widehat{\mathrm{Div}}(\mathcal{X})$ is called Arakelov's divisor on $\mathcal{X}$.

$\hat{D} = D + \alpha \cdot F_\infty$, $\alpha \in \mathbb{R}$ $= (D, \alpha)$

**Def.** $\widehat{D}_1, \widehat{D}_2 \in \widehat{\mathrm{Div}}(\mathfrak{X})$. We define $\widehat{D}_1 \cdot \widehat{D}_2$ as follows:

i) if $\widehat{D}_1 = D_1$ and $\widehat{D}_2 = D_2$ and $D_2$ vertical, then

$$\widehat{D}_1 \cdot \widehat{D}_2 = D_1 \cdot D_2 \quad \text{as defined before}$$

$$= \sum i_p(D_1, D_2)\, k(p)$$

ii) "$F_\infty$ behaves like a fibre" $\quad F_\infty \cdot F_\infty = 0$

iii) $D$ horizontal, $\quad D \cdot F_\infty = \deg_\eta D_\eta$

iv) $D_1, D_2 : B \to \mathfrak{X}$ sections, distinct

$$(D_1 \cdot D_2)_{Ar} = (D_1 \cdot D_2)_{\text{usual}} - gr_{\mathfrak{X}(\mathbb{C})}(D_{1,\mathbb{C}}, D_{2,\mathbb{C}})$$

**Def.** $f \in K(\mathfrak{X})^*$, $\widehat{\mathrm{div}}(f) = \mathrm{div}(f) + \underset{\underset{\widehat{\mathbb{R}}}{\cap}}{v_\infty(f)} \cdot F_\infty$

where $\displaystyle v_\infty(f) := -\int_{\mathfrak{X}(\mathbb{C})} \log|f|\, \mu_{Ar}$

**Lem.** $P : B \to \mathfrak{X}$, $f \in K(\mathfrak{X})^*$, $\widehat{P} = (P, 0)$

$$\widehat{P} \cdot \widehat{\mathrm{div}}(f) = 0$$

$$gr_X(\mathrm{div}(f), P) = \log|f|(P) + v_\infty(f)$$

$$\widehat{P} \cdot \widehat{\mathrm{div}}(f) = (P, \mathrm{div}(f))_{fin} + (P, \mathrm{div}(f))_{inf} + v_\infty(f) =$$

$$= \sum_{b \in B} v_b(f|_P) \cdot \log(\# k(b)) - gr_X(\mathrm{div}(f), P) + v_\infty(f)$$

$$= \sum_{b \in B} v_b(f|_P) \log(\# k(b)) - \log|f|(P) = \underbrace{0}_{\text{product formula}}$$

Thm (Arakelov). $\exists!$ bilinear symm intersection pairing

$$\widehat{\mathrm{Div}}(\mathcal{X}) \times \widehat{\mathrm{Div}}(\mathcal{X}) \longrightarrow \mathbb{R}$$

which respects "linear eq." $\quad \widehat{Cl}(\mathcal{X}) \times \widehat{Cl}(\mathcal{X}) \to \mathbb{R}$

Pf. Green's functions symm $\Rightarrow$ symmetry

Lemma $\Rightarrow$ respects linear equiv.

$\square$

Let $D$ divisor on $X$, $gr_X \rightsquigarrow$ metric on $\mathcal{O}_X(D)$

$1_D \in \mathcal{O}_X(D)$ canonical section, $\log \| 1_D \|^2_{Ar}(-) = gr_X(D, -)$

Ex: $P : B \to \mathcal{X}$ section, $g(\mathcal{X}) = 1$

$$\hat{P} \cdot \hat{P} = \text{constant} = -\frac{1}{12} \log \left| N_{K/\mathbb{Q}}(\Delta_{min}(\mathcal{X}_K)) \right|$$

Hriljac - Faltings - Hodge thm

$$\pi : \mathcal{X} \longrightarrow \mathrm{Spec}\, \mathcal{O}_K = B$$

Thm. The signature of $\langle -, - \rangle_{Ar} : \widehat{\mathrm{Div}}_{\mathbb{R}}(\mathcal{X}) \times \widehat{\mathrm{Div}}_{\mathbb{R}}(\mathcal{X}) \to \mathbb{R}$ is $(+, -, -, \ldots, -)$

Pf. If $D$ is an Arakelov divisor which is perpendicular to all fibers. Then $\deg D_\eta = 0$, and thus induces $\mathcal{O}(D) \in \mathrm{Jac}(X)$

Now

$$-\frac{1}{2[K:\mathbb{Q}]} (D, D)_{Ar} = \hat{h}_{NT}(\mathcal{O}(D))$$

Neron-T

using arithmetic Riemann-Roch

$\square$

Disadvantages of $\langle \cdot , - \rangle_{Ar}$

i) $g(X) \geq 1$ $\rightsquigarrow$ not natural!

ii) $gr_X$ $\rightsquigarrow$ there are other Green functions!

iii) $\mu_{Ar}$ is not compatible with pull-back nor push-forward

( $f_* \mu_{Ar}$ not even $C^\infty$ )

Deligne, Bost's extension of Arakelov's int. pairing

Deligne uses more Green functions

$\hat{\mathcal{Z}}_1(\mathcal{X}) = \{ (D, g) \mid D$ divisor, $g$ is an $L^2_1$-Green's function for $D \}$

Potential theory on $X$

$D$ divisor, $\mathrm{supp}(D) \subseteq \Omega \overset{\text{open}}{\subseteq} X$, $X \setminus \Omega$ not polar.

"Équilibrium potential" $gr_{D,\Omega}$ $\rightsquigarrow$ $\| \cdot \|^{cap}_{\Omega}$ capacitary metric.

Thm (Bost 199) Analogue of lemma 1:

$\hat{D}_\Omega \in \hat{\mathcal{Z}}_1(\mathcal{X})$ divisor s.t. $\hat{D}_\Omega \cdot \hat{D}_\Omega > 0$ and "..",

then $\mathrm{Supp}\, \hat{D}_\Omega$ is connected.

Pf. HIT for $\hat{\mathcal{Z}}_1(X)$. $\square$

Application: Let $\Omega \subseteq \mathcal{X}(\mathbb{C})$ s.t. ...

Thm (Bost-Chambert-Loir) $\mathcal{X}/\mathcal{O}_K$ arithmetic surface, $P \in \mathcal{X}(\mathcal{O}_K)$,

$\varphi \in \hat{\mathcal{O}}_{X,P}$ which is integral. Suppose $\varphi$ is algebraic over $\mathcal{O}_{X,P}$

If $\widehat{\deg}(T_P X, \|\cdot\|_{\Omega}^{cap}) > 0$, then $\varphi$ is a rational function on $K(\mathcal{X})$

From Schwarz lemma to Nevanlinna theory

**Thm 1** (Schwarz lemma) Let $f: \mathbb{D} \to \mathbb{D}$ holom., $f(0)=0$. Then $\forall z \in \mathbb{D}$, $|f(z)| \le |z|$, $|f'(0)| \le 1$

rescaling

**Thm 1'** Let $f: \mathbb{D}(r) \to \mathbb{D}(R)$ holom, $f(0)=0$. Then $|f(z)| \le \frac{R}{r}|z|$, $|f'(0)| \le R/r$.

Pf. wlog, $f$ extends a little beyond $\partial\mathbb{D}(r)$. (Then for gen. $f$, apply to $f|_{\mathbb{D}(r-\varepsilon)}$, $\varepsilon \to 0$).

Then $f(z) = z\,g(z)$, $g$ holom. Maximal modulus principle $\Rightarrow$

$$\Rightarrow \max_{|z|\le 1} |g(z)| = \max_{|z|=1} |g(z)| = \max_{|z|=1} = \max_{|z|=1} |f(z)| \le 1$$

Rem: $g(0) = f'(0)$

General statement. $f: \mathbb{B}^d(r) \to \mathbb{B}(R)$ hol, $\mathrm{mult}_0 f \ge i$, then $|f(z)| \le \left(\frac{\|z\|}{r}\right)^i R$

Pf. 1-dim. slice $\Rightarrow$ above argument to $z^i g(z) = f(z)$.

**Thm 2** (Liouville) $f: \mathbb{C} \to \mathbb{C}$ holom, bounded $\Rightarrow f = \text{const}$.

Pf: wlog $f(0)=0$, $|f| \le R$, let $z_0 \in \mathbb{C}$.

For $r > |z_0|$, apply thm 1' to $f|_{\mathbb{D}(r)} \Rightarrow |f(z_0)| \le \frac{R}{r}|z_0| \Rightarrow$ ($r \to \infty$)

$\Rightarrow f(z_0) = 0$

$\square$

**Def.** A complex sp. $X$ is a <u>Liouville sp</u> if every bounded holomorphic function is constant.

Eg: $\mathbb{C}$, $\mathbb{C}^n$, compact (without boundary?), $X(\mathbb{C})$ with $X$ algebraic variety.

$\mathbb{D}$, bounded domain in $\mathbb{C}^n$, are not

**Thm 3** (Picard) Let $f \in \mathcal{M}(\mathbb{C})$, missing at least 3 values in $\mathbb{P}^1(\mathbb{C})$.

Then it is constant.

Pf.

$$\begin{tikzcd} \mathbb{H} \arrow[d, "\lambda \text{ modular function}", "\text{universal cov.}"'] & & \mathbb{C} \arrow[ll, dashed] \arrow[dll, "f"] \\ \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\} & & \end{tikzcd}$$

upper half plane

$\tilde{f}: \mathbb{C} \to \mathbb{D}$ const $\Rightarrow$

$\Rightarrow f = $ const.

$\square$

Let $f \in \mathcal{M}(\mathbb{C})$

* Proximity functions: For $a \in \mathbb{P}^1(\mathbb{C})$, set

$$m_f(a, -): \mathbb{R}_{>0} \to \mathbb{R}, \quad r \longmapsto \begin{cases} \displaystyle\int_0^{2\pi} \log^+ |f(re^{i\vartheta})| \frac{d\vartheta}{2\pi}, & a = \infty \\ \displaystyle\int_0^{2\pi} \log^+ \left| \frac{1}{f(re^{i\vartheta}) - a} \right| \frac{d\vartheta}{2\pi}, & a \in \mathbb{C} \end{cases}$$

$\max\{\log, 0\}$

* Counting function

Let $\operatorname{ord}_z^+ f = \begin{cases} i & \text{if } f \text{ has a zero of ord. } i \\ 0 & \text{else} \end{cases}$

$$N_f(a, -): \mathbb{R}_{>0} \to \mathbb{R}: r \mapsto \operatorname{ord}_0^+ (f-a) \log r + \sum_{0 < |z| < r} \operatorname{ord}_z^+ (f-a) \log \frac{r}{|z|}$$

$$N_f(\infty, r) := N_{1/f}(0, r)$$

Thm 4 (Nevanlinna). For $f \in \mathcal{M}(\mathbb{C})$, $\exists\, T_f : \mathbb{R}_{>0} \to \mathbb{R}$, unique up to $\pm o(1)$, s.t. $\forall a \in \mathbb{P}^1(\mathbb{C})$, $m_f(a, r) + N_f(a, r) = T_f(r) + o(1)_{f, a}$

$T_f(r)$ is called the Nevanlinna height of $f$.

Thm (Chow): Every complex closed submanif of $\mathbb{P}^N(\mathbb{C})$ is alg.

Pf. 2 steps

1. Let $\varphi_1, \dots, \varphi_n : \mathbb{B}^d \hookrightarrow \mathbb{P}^N(\mathbb{C})$ holom emb, $V := \left(\varphi_1(\mathbb{B}^d) \cup \dots \cup \varphi_n(\mathbb{B}^d)\right)^{-\text{Zar}}$

For $i, D > 0$, set $E^i_D = \{ s \in \Gamma(V, \mathcal{O}(D)) \mid \mathrm{mult}_0(\varphi_v^* s) \geq i \ \forall v \}$

Assume $\exists\, c$ : if $\dfrac{i}{D} > c$, then $E^i_D = 0$.

Then $\dim V = d \Leftarrow$ each $\varphi_v(\mathbb{B}^d)$ is algebraizable

Step 2 Let $X \subseteq \mathbb{P}^N(\mathbb{C})$ closed ~~complex~~ of dim $d$

Find a finite atlas : $\varphi_v : \mathbb{B}^d \to \mathcal{U}_v \subset X$, set $P_v = \varphi_v(0)$

Let $L$ be an holom. l. b. on $\overline{X}^{\mathrm{Zar}}$, set

$$E^i_D = \{ s \in \Gamma(\overline{X}, L^{\otimes D}) \mid \mathrm{mult}_{P_v}(s) \geq i \quad \forall v \}$$

Then $\exists\, c > 0$, $\forall \ \frac{i}{D} > c$, $E^i_D = 0$

Indeed, choose

- $0 < r < r' < 1$ s.t. $\mathcal{U}_v(r) = \varphi_v(\mathbb{B}^d(r))$ still cover $X$.
- herm. metric on $L$, hence on $L^{\otimes D}$
- trivializations $\varepsilon_v \in \Gamma(\mathcal{U}_v, L)$

Let $s \in \Gamma(\overline{X}, L^{\otimes D}) \rightsquigarrow \exists$ hol. functions $f_v : \mathbb{B}^d \to \mathbb{C}$,

$s|_{\mathcal{U}_v} = (f_v \circ \varphi_v^{-1}) \cdot \varepsilon_v^D$

If $\mathrm{ord}_0 f_v = \mathrm{ord}_{P_v}(s) \geq i$, then Schwarz lemma $\Rightarrow$

$$|f_v(\varphi_v^{-1}(x))| \leq \left(\frac{r}{r'}\right)^i \sup_{w \in \mathbb{B}^d(r')} |f_v(w)| \quad \text{for all } x \in \mathcal{U}_v(r)$$

$$\|s(x)\| \leq \sup_{y \in \mathcal{U}_v(r)} \|\varepsilon_v(y)\|^D \cdot \underbrace{|f_v(\varphi_v^{-1}(x))|}$$

$$\leq \left(\frac{r}{r'}\right)^i \underbrace{\sup |f_v(w)|}$$

$$\leq \sup_{y \in \mathcal{U}_v(r')} \|\varepsilon_v(y)\|^{-D} \sup_{y \in \mathcal{U}_v(r')} \|s(y)\|$$

Hence, there are $\lambda, M$ $\quad 0 < \lambda < 1 < M$ s.t.

$$\max_{x\in X} \|s(x)\| \leq \underbrace{\lambda^i M^D}_{} \cdot \max_{x\in K} \|s(x)\|$$

if this is $<1$, then $s=0$ $\leadsto$ statement holds with $c = -\dfrac{\log M}{\log \lambda}$ $\square$

③ An estimate

$X \subseteq \mathbb{P}^N(\mathbb{C})$ closed manif., $L \to X$ line bdle, $P \in X$, $\mathcal{V}$ germ of analytic subsheaf of $X$ at $P$, $\dim \mathcal{V} > 0$.

$$E_D^i = \{ s \in \Gamma(X, L^{\otimes D}) \mid \mathrm{mult}_P(s|\mathcal{V}) \geq i \}$$

$i$-th jet maps ($\sim$ $i$-th partial derivatives)

$$\varphi_D^i : E_D^i \longrightarrow \mathrm{Sym}^i\, T_P \mathcal{V}^\vee \otimes L_P^{\otimes D}$$

Hermitian metric on $L$ $\leadsto$ operator norm $\|\varphi_D^i\|$

Thm 6 (Bost). Assume $\exists f : M \to X$ hol., $M$ Liouville manifold, $f(0) = P$, $f(\text{germ of } M \text{ at } 0) \Rightarrow \mathcal{V}$

Then, $\forall r > 0$, $\exists\, C(r) \in \mathbb{R}$ : $\|\varphi_D^i\| \leq \dfrac{C(r)^D}{r^i}$

Idea of pf: for $M = \mathbb{C}^d$,

trivialize $f^*L$, write for sections $s$: $f^*s = g \cdot \varepsilon^{\otimes D}$,

$$\|\varphi_D^i(s)\| \sim |g^{(i)}(0)| \cdot \|s\|$$

bound this by Schwarz

Rem. For $M = \mathbb{C}^n$, more precise estimate:

$$\|\varphi_D^i\| \leq \left(\frac{n\sqrt{e}}{r}\right)^i \left(e^{T_{f,\Sigma}(r)}\right)^D,$$ where $T_{f,\Sigma}$ is a generalized Nevalinna height.

About classical $T_f$:

$T_f(r) \gg \log r$ for $r \gg 0$

$T_f(r) = O(\log r) \iff f$ rational

Now $T_{f,L}(r) = o(\log r) \iff f$ regular map of alg varieties

---

Arithmetic algebraization à la Chudnovski:

Today, $\underbrace{3}_{\text{algebraizations}}$ thms about formal subschemes

$$\overset{3}{A} \Rightarrow \overset{1}{B} \Rightarrow \overset{2}{C}$$

about foliations

about algebraic and Lie groups.

cf [Bos01]

Thm B. Let $X$ be ~~a~~ sm. alg. variety over $k$ = # field, $P \in X(k)$.

$F \subseteq TX$ an involutive/integrable subvector bdle.

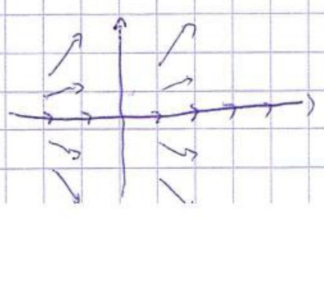
Then, the formal leaf of $F$ through $P$ is algebraic if the following conditions hold:

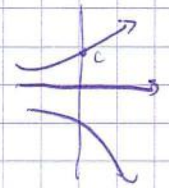
B1) Choose a model of $X, F$ over some open $U \subseteq \operatorname{Spec} \mathcal{O}_k$.

For almost all $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_k$, the subbdle $(F \bmod \mathfrak{p}) \subseteq T_{X_\mathfrak{p}}$ is stable under $p$-th powers $(p := \operatorname{char}(\mathcal{O}_k/\mathfrak{p}))$.

B2) For some $\sigma: k \hookrightarrow \mathbb{C}$, $\exists$ manif $M$, satisfying the Liouville property, $0 \in M$ and a meromorphic map $u: M \to X_\sigma(\mathbb{C})$, $u(0) = P_\sigma$ mapping a nbhd of $0$ bihol onto the germ of $F$ at $P$.

Ex. $X = \mathbb{A}^2_k = \operatorname{Spec} k[x,y]$ Let $F(x,y) = (1, a \cdot y)$, $a \in k$ fixed.



Characteristic lines:  $\gamma_c : z \mapsto (z, c\exp(az))$

$(\gamma_c)_{c\in\mathbb{C}}$ is the foliation associated with F.

Let $P=(0,1)$ complex leaf for $\sigma : k \to \mathbb{C}$ is the image of

$\mathbb{C} \xrightarrow{\gamma_1} \mathbb{A}^2_\sigma(\mathbb{C})$, $\gamma_1(z) = (z, 1\cdot \exp(\sigma(a)z))$.

↳ This gives $\boxed{B2}$

Check that F mod $\mathfrak{p}$ is stable under p-th power.

As a derivation, F (abuse of notation) is $\partial_F : f(x,y) \mapsto \frac{\partial f}{\partial x}(x,y) + ay\frac{\partial f}{\partial y}(x,y)$

Now mod $\mathfrak{p}$, $\partial_F^p : x \mapsto 0$, $y \mapsto a^p y$

Need $\partial_F^p = \lambda_p \partial_F$, $(0, a^p y) = \lambda_p (1, ay)$

Need $\underset{\substack{\shortparallel \\ a}}{a^p} = 0$ mod $\mathfrak{p}$ ~~for~~ for ~~all~~ almost all $\mathfrak{p}$ $\Leftrightarrow$ $a=0$

<u>Rem</u>. Hence, $\exp(az)$ is algebraic $\Leftrightarrow$ $a=0$.

<u>Thm C</u>. Let $G$ be an algebraic gp over $k$, let $\mathfrak{h}$ be a subalg. of $\mathfrak{g} = \mathrm{Lie}(G)$. There exists an algebr. subgp $H < G$ with $\mathrm{Lie}(H) = \mathfrak{h}$ if the following hold

$\boxed{C1}$ ($\mathfrak{h}$ mod $\mathfrak{p}$) $\subset$ ($\mathfrak{g}$ mod $\mathfrak{p}$) is stable under p-th power for almost all $\mathfrak{p} \subset \mathcal{O}_k$.

$\boxed{C2}$ 6

<u>Rem</u>. The converse is also true

Rem C2 $\phi$ because $\underbrace{\phantom{A}}_{B2}$ is $\overbrace{\text{automatically}}^{\text{somehow}}$ fulfilled

Ex: $G = \mathbb{G}_m \times \mathbb{G}_m$. $\mathrm{Lie}(G) = \langle x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y} \rangle$

$\cup$

$h$ given by $\langle \underbrace{x\frac{\partial}{\partial x} + a y \frac{\partial}{\partial y}}_{u_a} \rangle$, $a \in k$ fixed

Compute $p$-th powers mod $\mathfrak{p}$

$$(u_a)^p = \begin{cases} x \longmapsto x \\ y \longmapsto a^p y \end{cases} \qquad (u_a)^p \in h \bmod \mathfrak{p} \iff a \equiv a^p \bmod \mathfrak{p}$$

The Lie-subalgebra $h_a$ is algebraic $\Leftrightarrow$ $a^p \equiv a \bmod \mathfrak{p}$, then C

$\Longleftrightarrow$ $\mathbb{C} \to \mathbb{C}^* \times \mathbb{C}^*$; $z \mapsto (\exp(z), \exp(\sigma(a) z))$ is algebraic $\Leftrightarrow$

$\Leftrightarrow \sigma(a) \in \mathbb{Q}$ (Kronecker 1880).

## Sizes of formal subschemes

$K/\mathbb{Q}_p$ finite field ext, $X/K$ alg. variety, $P \in X(K)$.
$\hat{V}$ a formal smooth subscheme of $\hat{X}_P$.

Suppose $\hat{V}$ is the image of a map $\hat{\mathbb{A}}^{v}_K \longrightarrow \hat{\mathbb{A}}^{d}_K$, $v \leq d$

To give a map between these formal schemes is equivalent to give a map of the rings. $u_1, \dots, u_d$

$\hat{V}$ large $\Longleftrightarrow$ $u_1, \dots, u_d$ large convergence radius.

Notation: for $g \in K[[x_1, \dots, x_d]]$, say $g = \sum_{I \in \mathbb{N}^d} a_I x^I$, set $r \geq 0$

$$\|g\|_r := \sup_{I \in \mathbb{N}^d} |a_I| \cdot r^{|I|} \in [0, \infty]$$

Automorphisms of $\widehat{\mathbb{A}}^d_K$ are given by $d$ series $(g_1, -, g_d) = g$ of in $d$ variables, $g(0) = 0$

$$(Dg)(0) \in GL_d(K)$$

For $r > 0$, let $G(r) \subseteq Aut(\widehat{\mathbb{A}^d_K})$ be the subgp formed by $g = (g_1, -, g_d)$ s.t. $Dg(0) \in GL_d(\mathcal{O}_K)$

$$\|g_i\|_r \leq r$$

$G(r)$ sends the disc of radius $r$ in $K^d$ to itself.

$r' \geq r \Rightarrow G(r) \supseteq G(r')$, $G(0)$ series

$Dg(0) \in GL_d(\mathcal{O}_k)$ positive conv. radius

Let $\hat{V}$ a smooth formal subsch. of $\widehat{\mathbb{A}^d_K}$. There exists an element $g$ of $G(0)$ s.t. $g^*(\hat{V}) = \widehat{\mathbb{A}^\nu_K} \times \{0\} \subseteq \widehat{\mathbb{A}^d_K}$

Def. The <u>size</u> of $\hat{V}$ is

$$R(\hat{V}) := \sup\{ r \in [0,1] \mid \exists g \in G(r) \text{ with } g^*(\hat{V}) = \widehat{\mathbb{A}^\nu_k} \times 0 \}$$

Extend this def. to $\hat{V} \subseteq \widehat{X_p}$ by choosing $\mathcal{X}/\mathcal{O}_K$ and an embedding $\mathcal{X} \hookrightarrow \widehat{\mathbb{A}^d_{\mathcal{O}_k}}$

<u>Thm A</u> $k$ # field, $X/k$ alg. variety, $p \in X(k)$, $\hat{V}$ be a smooth formal subscheme of $\widehat{X_p}$. There exists an alg. subvariety $Y \subseteq X$, $p \in Y(k)$ s.t. $\hat{V}$ is a branch of $Y$ at $p$, <u>if</u> the following holds

$\boxed{A1}$ For every $\mathfrak{p} \subseteq \mathcal{O}_k$ maximal, the size $R_\mathfrak{p}(\hat{V})$ is $> 0$ and the sum

$$\sum_{\text{almost all } \mathfrak{p}} -\log(R_\mathfrak{p}(\hat{V})) \quad \text{is finite}$$

$\boxed{A2}$ For every $\sigma: k \hookrightarrow \mathbb{C}$, the formal subscheme $\hat{V} \subseteq X_\sigma(\mathbb{C})$ is the germ of an analytic submanifold, and for over $\sigma_0 : k \hookrightarrow \mathbb{C}$ $\exists$ complex manifold $M$ with Liouville property, meromorphic $u : M \to X_{\sigma_0}(\mathbb{C})$, biholom. onto the analytic germ of $\hat{V}$ in $X_{\sigma_0}(\mathbb{C})$.

Ideas on the proof:

- Assume $\hat{V}$ is a germ of a curve (this avoids Sym. powers)
- Assume $X$ projective, comes with ample line bdle $L$.
  $\hat{V}$ is Zariski-dense in $X$.
- $k = \mathbb{Q}$, so that $\widehat{\deg} = \widehat{\deg}_n$

Evaluation maps: $D \geq 0$, $E_D = \Gamma(X, L^{\otimes D})$

$$\eta_D^i : E_D = \Gamma(X, L^{\otimes D}) \longrightarrow \Gamma(V_i, L^{\otimes D})$$

$$E_D^i = \ker \eta_D^{i-1}$$

Obtain a filtration on $E_D$

$$E_D^i \longrightarrow \ker\left(\Gamma(V_i, L^{\otimes D}) \to \Gamma(V_{i-1}, L^{\otimes D})\right)$$

$$\| $$

$$T_V^{\otimes -i} \otimes L_p^{\otimes D}$$

(← previous talk $\mathrm{Sym}^i$ $^\vee$ because $V$ was not 1-dim)

<u>Lem.</u> If $\hat{V}$ is not algebraic, then

$$\lim_{D \to \infty} \sum_{i \geq 0} \frac{i}{D} \frac{\mathrm{rk}(E_D^i / E_D^{i-1})}{\mathrm{rk}(E_D)} = +\infty$$

<u>Choose</u>

- Model $\mathcal{X}, \mathcal{L}, \mathcal{P}$ of $X, L, p$ over $\mathrm{Spec}\, \mathcal{O}_K$ (here $\mathbb{Z}$).
  and ~~an~~ hermitian metric on $\mathcal{L}$.
- A positive Lebesgue measure on $X(\mathbb{C})$

. A norm $\|\cdot\|_0$ on the tangent line $t = T_{\hat{V}, P}$

$$\tilde{t} \simeq (\mathbb{Z}, \|\cdot\|_0)$$

Input from analysis: on $T_{\hat{V}, P, \mathbb{C}} = t_{\mathbb{C}}$, define the <u>canonical seminorm</u>

$$\|\cdot\|_{can} = \exp\left(\limsup_{i/D \to \infty} \frac{1}{i} \log\left(\underbrace{\|\gamma_D^i\|}_{\text{operator norm of } \gamma_D^i}\right)\right) \cdot \|\cdot\|_0$$

$\gamma_D^i : E_D^i \to t^{\otimes -i} \otimes L_P^{\otimes D}$

<u>Aside</u>: $\widehat{\deg}(t, R_p^{-1} |\cdot|_p, \|\cdot\|_{can}) > 0$

$\hookrightarrow$ Analog of Andreotti Hartshorne.

Hypothesis A2: there exist $\lambda > 0$, $d > 0$ s.t. for all $(D, i)$, $i > \lambda D$.

$$-\frac{1}{i} \log \|\gamma_D^i\| \geq d \underbrace{- \sum_p \log R_p(\hat{V}) - \widehat{\deg}\, t}$$

Slopes: $-cD \leq \hat{\mu}(\bar{E}_D) \leq \frac{1}{rk\, E_D} \sum_{i \geq 0} rk\left(E_D^i / E_D^{i+1}\right)\left[\widehat{\deg}\left(\bar{t}^{\otimes -i} \otimes \mathcal{L}_p^{\otimes D}\right) + ht(\gamma_D^i)\right]$

$L^2$-norm

$$\widehat{\deg}\left(\bar{t}^{\otimes -i} \otimes \mathcal{L}_p^{\otimes D}\right) = -i\, \widehat{\deg}(\bar{t}) + D \cdot \widehat{\deg}(\mathcal{L}_p)$$

$$\text{height}(\gamma_D^i) = \sum_p \underbrace{\log(\|\gamma_D^i \otimes \mathbb{Q}_p\|)}_{\leq R_p(\hat{V})^{-i}} + \underbrace{\log(\|\gamma_D^i \otimes_{\mathbb{Q}} \mathbb{C}\|)}_{\leq \alpha i + \beta D}$$

for some $\alpha, \beta \geq 0$

$\vdots$

$\square$

## Applications

### §1. Isogeny between elliptic curves

Let $E/\mathbb{Q}$ ell. curve, with good reduction over $\operatorname{Spec} \mathbb{Z}[1/N]$

$\mathcal{E} \to \operatorname{Spec} \mathbb{Z}[1/N]$ its regular model.

(Hasse Weil)
Fact The $p$-th power map on $\operatorname{Lie} \mathcal{E}/\mathbb{F}_p$ is the multiplication by

$$a_p(E) = p+1 - |\mathcal{E}(\mathbb{F}_p)|$$

Thm. For any $E, E'/\mathbb{Q}$ elliptic curves, TFAE

1) isogeneous $/\mathbb{Q}$
2) $a_p(E) = a_p(E')$ for almost all $p$.

Pf. $G = E \times E'$. To give a $\mathbb{Q}$-isogeny is equiv. to give a subgp $H \subsetneq \underbrace{G}_{\text{of dim } 1}$ surjecting to both factors

(true for abelian varieties, $\dim H = \dim A$)

$\mathfrak{h} \subseteq \operatorname{Lie} G = \operatorname{Lie} E \oplus \operatorname{Lie} E'$ of dim 1, distinct from $\operatorname{Lie} E \oplus \{0\}$, $0 \oplus \operatorname{Lie} E'$

Let $N \in \mathbb{N}$ s.t. $\mathcal{E}, \mathcal{E}' \to \operatorname{Spec} \mathbb{Z}[1/N]$ regular models of $E, E'$.

Then $a_p(E) = a_p(E')$, $p \gg 0$

$|a_p(E)| \leq 2\sqrt{p}$ $\Updownarrow$

$a_p(E) = a_p(E') \mod p \iff$

$\iff \mathfrak{h}$ is stable under $p$-th power $\iff$

$\iff \mathfrak{h}$ is alg $\iff E \sim E'$ isogeny over $\mathbb{Q}$ $\square$

Thm B

## § Groth. - Katz conjecture

### 1_ Background diff equations

Consider the linear diff system

$$\frac{d}{dz} Y = A(z) Y, \qquad A(z) \in M_d(\mathbb{Q}(z))$$

Goal: give criterion for this system to have a basis of alg. solutions, i.e. the holomorphic solutions on a nbhd of $z_0 \in \mathbb{Q} \setminus \{\text{poles of } A(z)\}$ are algebraic over $\mathbb{Q}(z)$.

Conjecturally, criterion is:

$$(*) \begin{cases} \text{for almost all } p, \quad \frac{d}{dz} Y = A(z) Y \mod p \\ \text{has a basis of algebraic solutions over } \mathbb{F}_p(z) \end{cases}$$

Rem: $(*)$ can be verified

Lemma: Define
$$\begin{cases} A_0(z) = Id \\ A_1(z) = A(z) \\ \vdots \\ A_{n+1}(z) = \frac{d}{dz} A_n(z) + A_n(z) \cdot A(z) \end{cases}$$

Then TFAE

1) $\frac{d}{dz} Y = A(z) Y$ mod $p$ has a basis of algebraic solutions over $\mathbb{F}_p(z)$

2) " solutions in $\mathbb{F}_p(z)$

3) " solutions in $\mathbb{F}_p((z))$

4) $A_p \equiv 0 \mod p$

"p-curvature"

Pf. 2) ⇒ 1) ⇒ 3) ✓

3) ⇒ 4) $Y(z)$ solution ⇒ $\frac{d^n}{dz^n} Y(z) = A_n(z) Y(z)$

$Y(z)$ is a solution in $\mathbb{F}_p((z)) \Rightarrow \frac{d^p}{dz^p} Y(z) = 0 \mod p \Rightarrow$

$\Rightarrow A_p(z) = 0 \mod p$

4) $\Rightarrow$ 1) $Y(z) = \left( \sum \frac{(-z)^i}{i!} A_i(z) \right)^{-1}$ is a basis of solutions in $\mathbb{F}_p(z)$

$\square$

Statement of Groth.-Katz conjecture

Conj.: $X$ = smooth quasi-proj. var. $/K$ = # field

$(E, \nabla)$ = v.b. $/X$ ~~with~~ integrable connection

If for almost all $p$ the $p$-curvature of $(E,\nabla)$ vanishes, then

$\exists$ finite étale $f: Y \to X$ s.t. $f^*E$ = trivial

Explanation: $E$ v.b. $/X$, $X/K$, $\mathcal{E}$ = sheaf of sections of $E$

- Connection: $\nabla: \mathcal{E} \longrightarrow \Omega^1_X(\mathcal{E})$ $K$-linear s.t.

$\forall\, \mathcal{U} \hookrightarrow X$, $e \in \Gamma(\mathcal{U}, \mathcal{E})$, $f \in \mathcal{O}_X(\mathcal{U})$, $\rightsquigarrow$ $\nabla(fe) = df \cdot e + f\nabla e$

Rem 1) $\nabla_2 - \nabla_1$ is $\mathcal{O}_X$-linear (by Leibniz rule)

$\nabla_0$ connection, $\gamma \in \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \Omega^1_X(\mathcal{E})) \Rightarrow \nabla_0 + \gamma$ connection

$\rightsquigarrow$ {connections on $E$} is a ppal homog. space under $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \Omega^1_X(\mathcal{E}))$ (maybe $\emptyset$).

2) We can extend $\nabla$ to $\nabla_i : \Omega^i_X(\mathcal{E}) \to \Omega^{i+1}_X(\mathcal{E})$

Fact $\left\{ \begin{matrix} \text{flat} \\ (E, \nabla) \\ + \text{regular singularities} \end{matrix} \right\} \xleftrightarrow{\text{equiv}} \left\{ \text{Repr. of } \pi_1(X, x) \atop \downarrow \atop \mathrm{Aut}(E_x) \right\}$

- Integrability: Given $(E,\nabla)$, we have a map

$TX \longrightarrow \mathcal{E}nd(\mathcal{E})$, $\partial \mapsto \nabla(\partial)$

$$\begin{array}{c} \mathcal{E} \xrightarrow{\nabla} \Omega^1_X(\mathcal{E}) \\ \qquad\quad \downarrow \partial \otimes 1 \\ \qquad\quad \mathcal{E} \end{array}$$

Def. The curvature of $\nabla$ is

$$\Psi(E,\nabla) := \nabla^2$$

Rem. $\nabla^2$ is always $\mathcal{O}_X$-linear, so $\nabla^2 \in \mathcal{E}nd(\mathcal{E}) \otimes_{\mathcal{O}_X} \Omega^2_X$

Prop TFAE

1) $TX \to \mathcal{E}nd(\mathcal{E})$, $[\nabla(\partial_1), \nabla(\partial_2)] = \nabla([\partial_1, \partial_2])$
2) $\Psi(E,\nabla) = 0$

Pf. $\Psi(E,\nabla)(\partial_1, \partial_2) = [\nabla(\partial_1), \nabla(\partial_2)] - \nabla([\partial_1,\partial_2])$ (cf Deligne) $\square$

p-curvature

char $K = p > 0$

$(E,\nabla)$ is integrable

Def. The p-curvature $\Psi_p(E,\nabla): TX \to \mathcal{E}nd(\mathcal{E})$, $\partial \mapsto \nabla(\partial)^p - \nabla(\partial^p)$

Rem. $\Psi_p(E,\nabla)(f_1\partial_1 + f_2\partial_2) = f_1^p \Psi_p(E,\nabla)(\partial_1) + f_2^p \Psi_p(E,\nabla)(\partial_2)$

Thm ((Cartier). TFAE

1) $\Psi_p = 0$
2) $TX \to \mathcal{E}nd(\mathcal{E})$, $\partial \mapsto \nabla(\partial)$ is a homom. of p-Lie alg.
3) $(E,\nabla)$ is trivial, i.e. $\mathcal{E}$ is gen. by $\mathcal{E}^\nabla = \{e \mid \nabla e = 0\}$ as $\mathcal{O}_X$-mod.

3 - André's thm $K = \#$field

Prop. $(E,\nabla)/X$ integrable connection. $\langle E,\nabla\rangle^\otimes :=$ category of $\nabla$-stable subquotients of all the tensor products $E^{\otimes m} \otimes E^{\vee \otimes n}$

$X$ connected $\Rightarrow \langle E, \nabla \rangle^{\otimes}$ is Tannakian

<u>Def.</u> The <u>geometric Galois differential gp</u> $G$ of $(E, \nabla)$ is the subgp of $GL(E \otimes \kappa(X))$ stabilizing all objects of $\langle E, \nabla\rangle^{\otimes}$

<u>Rem.</u> $(E, \nabla)$ of regular singularities.

$\forall K \hookrightarrow \mathbb{C}, \quad (E^{an}, \nabla^{an}) / X(\mathbb{C})$

Galois diff. gp = Zariski closure of $\mathrm{im}\left(\pi_1(X(\mathbb{C}), x) \to GL(E_x)\right)$

<u>Thm</u> (André) The Grothendieck-Katz conj. holds if the Galois diff. gp of $(E, \nabla)$ is virtually solvable.

<u>Pf</u> (Sketch)

<u>Step 1</u> Reduction to the case of $\dim X = 1$, because of

$$\begin{tikzcd} \pi_1(C, x) \arrow[r, two heads] & \pi_1(X, x) \arrow[d] \\ & GL(E_x) \end{tikzcd}$$

$\exists\, C \hookrightarrow X$ curve $\forall x \in C$

Problem: Liouville property fails!

<u>Step 2</u> Replace $X$ by $S^k X$ symmetric power $\leftarrow G$ is solvable

Dévissage $\rightsquigarrow$ may assume $G$ is connected, commutative

$E^k \longrightarrow X^k$ $\quad$ sum : $G^k \to G$

$\hookrightarrow G^k$-bdle

$$\mathrm{sum}_* E^k / \sigma_k \longrightarrow X^k / \sigma_k$$

$\sigma_k$ $\hookrightarrow$ permutations

<u>Step 3</u> For $k \gg 0$, Liouville property holds.

$S^k X$ "looks like" $\mathrm{Pic}_D(\bar{X})$, $\bar{X}$ compactification

$\hookrightarrow$ birrational

§ 3.

<u>Thm</u>. $X$ = geom connected sm. proj. curve / $K$

$P \in X(K)$, $\varphi \in \widehat{\mathcal{O}}_{X,P}$ formal function around $P$

s.t. the formal graph of $\varphi$ in $\widehat{X \times \mathbb{A}^1}_{(P, \varphi(P))}$ is $A$-analytic

$\Omega \in X(\mathbb{C})$ with same property

If $\widehat{\deg}(T_P X, \|\cdot\|^{cap}_{\Omega}) > 0$, then $\varphi$ is the formal germ at $P$ of a rational function on $X$.

{ Avilgau (alg. closed)
{ Peter (algebraization)

Easy Case of Groth - Katz

<u>Ex</u>: $X = \mathbb{G}_m / K$, $E = \mathcal{O}_X$, $\nabla$ is given by image of 1, hence

Let $\nabla(1) := a \frac{dx}{x}$, $a \in K$.

then $\pi_1^{top}(X, *) = \mathbb{Z} \longrightarrow \mathbb{C}^\times$

$1 \longmapsto \exp(-2\pi i \cdot a)$

$a^p - a = 0 \iff a$ rational, as seen this morning.

Then, consider an étale cover $f$ of degree = denominator

Then $f^*(\mathcal{O}_X, \nabla)$ is trivial!! :)

=

<u>Overview</u>

<u>Thm 1</u> $K$ = # field, $\sigma_0 : K \to \mathbb{C}$, ① $f_1, \dots, f_N$ mer. functions of finite growth $\le \rho$, (i.e. $f_i = g_i / g_{N+i}$, $g_i$ entire, $\log |g_i(z)| \le |z|^\rho$ for $|z| \gg 1$).

② $\operatorname{tr} \deg_K K(f_1, \dots, f_N) \ge 2$

③ diff eq. $\frac{d}{dz} \subset K[f_1, \dots, f_N]$.

then, $\# \{ z \in \mathbb{C} \setminus \{poles\} \mid f_i(z) \in K \} \le 20\rho \cdot [K : \mathbb{Q}]$

Rem. One can relax ③ by a local one

Thm 2 (Gasbarri '10) $\underbrace{X}_{sm}$ quasi-proj. variety $/\underbrace{K,}_{\dim\geq 2}$ $p : \mathbb{C} \to X(\mathbb{C})$ holom. map s.t.

① finite growth (i.e. $i: X \hookrightarrow \mathbb{P}^k$, $(i \circ p)(z) = (\varphi_i(z))$ meromorphic of growth $\leq \rho$

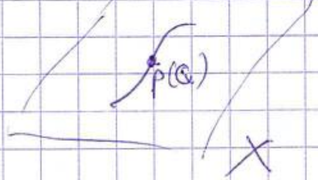
② $p(\mathbb{C})$ has Zariski dense image

Then, $\# \{ Q \in \mathbb{C} \,/\, p(Q) \in X(K)$ "$\alpha$-good" (wrt $p$) $\} \leq \dfrac{\dim X + 1}{\dim X - 1} \cdot \rho\, \alpha\, [K:\mathbb{Q}]$

For any $Q$,

$p$ induces

$$\hat{p}_{Q,\mathbb{C}} : \hat{\mathbb{A}}^1_{\mathbb{C}} = \mathrm{Spf}\left(\mathbb{C}[[T]]\right) \to \hat{V}_{Q,\mathbb{C}} \subseteq \hat{X}_{P,\mathbb{C}}$$




Roughly, $Q$ is "$\alpha$-good" if $\hat{p}_{Q,\mathbb{C}}$ comes (after some coord.-change) from a base change from something over $K$

i.e. it should exist $S$ = set of places $\supset \Sigma_{K,\infty}$

1) $\exists$ models $\mathcal{X} \to (\mathcal{O}_K \setminus S)$, $p(Q) = P$ has a ext. $\mathcal{O}_K \setminus S \to \mathcal{X}$

2)
⋮
more conditions

Cor. $F$ line bdle $\subseteq$ tgt bdle $(X)$

$\mathcal{F}$ analytic integral curve of $F_{\mathbb{C}}$

$p : \mathbb{C} \to \mathcal{F}$ étale, holom. of finite growth

Then, $\# \{ Q \in \mathbb{C} \mid p(Q) \in X(K) \} \leq 3\, \rho\, [K:\mathbb{Q}]$

or

$\mathcal{F}$ is alg. curve (every pt is "1-good")

## Algebraization à la Schneider-Lang II

**Version**: Let $X/K$ sm variety, $K \overset{\hookrightarrow \mathbb{C}}{=} \#\text{field}$, $\mathcal{F}$ a 1-dim. foliation, $p \in X(K)$, $\tilde{V}_p$ complex leaf through $p$.
Here no more assumptions on $\mathcal{F}$.

Assume we have

$$\varphi : \mathbb{C} \longrightarrow \tilde{V}_p \quad \text{meromorphic } \underline{\text{of order} \leq \rho}$$

Let $F \subset \mathbb{C} \setminus \{\text{poles}\}$, s.t. $\varphi(F) \subset X(K)$

If $\dfrac{\# F}{\rho \cdot [K:\mathbb{Q}]} \geq C$ (absolute constant), then $\tilde{V}_p$ is algebraizable.

Rem. If $F$ is infinite, then $\tilde{V}_p$ is algebraizable.

Cor (Hermite-Lindemann). If $\alpha \in \overline{\mathbb{Q}}^*$, then $\exp(\alpha)$ is trascendental

Pf. The graph $\Gamma$ of $\exp : \mathbb{C} \to \mathbb{C}$ is the leaf of a foliation on $\mathbb{C}^2$.

If $(\alpha, \exp(\alpha)) \in K^2$ # field, $(n\alpha, \exp(n\alpha)) \in K^2 \Rightarrow$

$\rightsquigarrow$ we get infinitely many pts in $\Gamma \cap K^2 \Rightarrow$ algebraizable $\lightning$

□

Strategy (F finite). Consider $\hat{V}_F$ the formal scheme with support in $\varphi(F)$ defined by $F$ (formal $K$-scheme). Now consider $\mathcal{L}$ hermitian ample line bdle on $\mathcal{X}$, a model of $X$ over $\mathcal{O}_K$.

Consider the evaluation map

$$ev : H^0(\mathcal{X}, \mathcal{L}^D) \longrightarrow H^0(\hat{V}, \mathcal{L}^D)$$

lim of $H^0$ over thickenings (K-vector spaces)

If $ev$ is not inj., then $V$ is not Zariski dense.

(Assume $\dim X = 2$) Then $ev$ not inj $\Rightarrow$ $V$ algebraizable

Need to put metrics + integral structure on RHS. Then show that it is not too high.

→ Integral structure.

We need an int. str. on $H^0(\hat{V}, \mathcal{L}^D)$ making ev integral

$$= \varprojlim H^0(V_i, \mathcal{L}^D)$$

Note that $X/\mathcal{O}_K \Rightarrow$ we can make an int. structure, but this is not the right one!

Somehow with the exponential we get problems.

Put the structure s.t. $s$ is a section of $\mathcal{L}^D$ that vanishes to order $k$. at a pt $p$. $k!\, j^k(s) \in S^k N_p \hat{V} \otimes \mathcal{L}^D$ is integral

(we put $k!$ to survive the derivatives).

→ Metric: take the quotient metric (on the image of $H^0(\mathcal{X}, \mathcal{L}^D)$).

(Another option: metric on $S^k N_p \hat{V} \otimes \mathcal{L}^D$ and then something else)

Let $E_i^D$ be the hermitian $\mathcal{O}_K$-module corresponding to sections of $\mathcal{L}^D$ on $\hat{V}$ ~~up to sections~~ vanishing to the ~~ith~~ order on the support. $i-1$-th

Let $L_i^D$ be the quot. $E_i^D / E_{i+1}^D$ ⇝ this is an hermitian line bdle on $\mathcal{O}_K$. Can define $h^0_\theta(L_i^D)$

If $\sum_{i \geq 0} h^0_\theta(L_i^D) < +\infty$, then we can define $h^0_\theta(H^0_{int}(\hat{V}, \bar{\mathcal{L}}^D))$
$\underset{= L_i^D(\epsilon)}{}$

so that (i) $h^0_\theta(H^0_{int}(\hat{V}, \bar{\mathcal{L}}^D)) \leq \sum_i h^0_\theta(L_i^D)$

(ii) If $ev: H^0(\mathcal{X}, \bar{\mathcal{L}}^D) \to H^0_{int}(\hat{V}, \bar{\mathcal{L}}^D)$ injective, then

$$cD^3 \leq h^0_\theta(H^0(\mathcal{X}, \bar{\mathcal{L}}^D)) \leq h^0_\theta(H^0_{int}(\hat{V}, \bar{\mathcal{L}}^D))$$

arithmetic Hilbert-Samuel

Summarising inequalities of the week

| | Geom case | Bost-Chudnovski | Serge-Lang |
|---|---|---|---|
| $h^0_\theta(L^D_i) \le$ | $(-\alpha i+\beta D)^+$ <br> $\max\{-,0\}$ | $\alpha i+\beta D$ | $i\cdot\log(i)+\beta D$ |
| Schwarz lemma <br> $i>\lambda D$ <br> $h^0_\theta(L^D_i)\le$ | — | $(-\gamma i+\beta D)^+$ | $\left(i\log(i)+\beta D-\gamma\frac{\#F}{\rho}\, i\log\frac{i}{D}\right)^+$ |
| $h^0_\theta(L^D_i)=0$ <br> for $i=\dots$ | $i=\lambda D$ | $i=\lambda D$ | $i=D^{1+\frac{1}{\gamma\frac{\#F}{\rho}-1}}$ |
| $h^0_\theta\approx$ | $\approx D^2$ | $\approx D^2$ | $\approx D^{2+\frac{1}{\gamma\#F/\rho-1}}$ |

Proof of

Applications

- "Easy" higher dimensional variant.
- To Lie groups. $K=\#$ field, $G/K$ commutative algebraic gp. $H\subset \operatorname{Lie} G$ sub $K$-Lie alg.

Assume $H_{\mathbb{C}}$ has a basis $B$ s.t. $\exp(B)\subset G(\overline{\mathbb{Q}})$. Then $H$ is algebraic.

E.g. (Gelfand-Schneider) Apply to $\mathbb{G}_m\times\mathbb{G}_m$

If $\alpha\in\overline{\mathbb{Q}}^*$, $\alpha\neq 1$, $\beta\in\overline{\mathbb{Q}}$, $\alpha^\beta\in\overline{\mathbb{Q}}\Rightarrow\beta\in\mathbb{Q}$

Ex. $G_1, G_2$ comm. alg. groups same dim

Assume that $\mathrm{Lie}(G_1)_{\mathbb{C}}$ is generated as $\mathbb{C}$-vect. sp. by periods

Consider a morphism

$$\mathrm{Lie}\, G_1 \xrightarrow{\phi} \mathrm{Lie}\, G_2 \quad \text{s.t.}$$

$\phi_{\mathbb{C}}$ (periods of $G_1$) $\subset$ periods of $G_2$, Then $\boxed{\phi \text{ comes from } G_1 \to G_2}$

$\ker \exp_{G_1}$

E.g. $E_1, E_2 / K$ elliptic curves

$$H^1_{dR}(E_1/K) \xrightarrow{\phi} H^1_{dR}(E_2/K)$$

$$\phi_{\mathbb{C}} : H^1_{sing}(E_1, \mathbb{C}) \to H^1_{sing}(E_2, \mathbb{C})$$

$$H^1_{sing}(E_1, \mathbb{Q}) \to H^1_{sing}(E_2, \mathbb{Q})$$

If $\phi_{\mathbb{C}}(H^1_{sing}(E_1, \mathbb{Q})) \subset H^1_{sing}(E_2, \mathbb{Q})$, then $\phi$ comes from an isogeny.

## Cycles in $H^{\otimes}_{dR}(A/K)$

abelian varieties

Let $K = \#$ field, $A$ polarized ab. variety, $v$ place of $K$

$H^i_{dR}(A/K)$, $\quad H^{\otimes}_{dR}(A/K) := \bigoplus^{2\dim A} H^i_{dR}(A/K)$ with tensor alg. structure

I. An application of algebraization thm.

Thm (Bost): $G/K$ comm alg gp, $W \subset \mathrm{Lie}\, G$ $K$-v.s.
Assume for almost all $v$ of $K$, $W \otimes k_v$ is closed under p-th power map.

$k_v$ res. field

Then $\exists H \subseteq G$ alg. s.t. $\mathrm{Lie}\, H = W$

Rem. Gasbarri and Herblot. $\exists M$ a set of prime numbers s.t.

s.t. $\forall p \in M$, $\forall v | p$, $W \otimes k_v$ is closed under $p$-th power $\Rightarrow$

$\Rightarrow W$ is algebraic.

Cor (Bost) Assume $s \in \mathrm{End}_K(H^1_{dR}(A/K))$ is fixed by the crystalline Frob $\varphi_v$ for almost all $v$ (a density one - analytic number theory sense - is enough). Then $s$ is algebraic.

Pf. $H^1_{dR}(A/K) = \mathrm{Lie}\ E(A^\vee)$, $\varphi_v(s) = s$

$\varphi_v \otimes k_v = p$-th power map $\Leftrightarrow \Gamma_s$ is closed under $p$-th power

□

II Ogus conjecture

setting: $L/K$ fin. ext, for almost all $v$. $\varphi_v \curvearrowright H^i_{dR}(A_L/L) \otimes L_v$

Conj (Ogus): Let $s \in H^{\otimes}_{dR}(A_L/L)$. If $\varphi_v(s) = s$ for almost all $v$, then $s$ is a Hodge cycle.

Hodge cycles: Fix $\sigma: K \hookrightarrow \mathbb{C}$, $s \in H^{\otimes}_B(A_\sigma(\mathbb{C}), \mathbb{Q})$ and $s \in (\ )^{0,0} / \Leftrightarrow$

$(\Leftrightarrow s \in \mathrm{Fil}^0)$

Rem 1) The converse statement $\to$ Hodge is fixed by $\varphi_v$ for $\forall v$ (Ogus / Deligne + Blasius)

2) Cor verified the conj. for $s \in \mathrm{End}(H^1_{dR}(-))$

3) Ogus: conj for $A$ has CM

Serre-Tate (André's book) $A =$ ell. curves

III de Rham-Tate cycles + main result.

dR-T: $s \in H^{\otimes}_{dR}(A_L/L)$, $\varphi_v(s) = s$ $\forall\forall v$, and $\forall \sigma: K \hookrightarrow \mathbb{C}$, via dR-B comparison, $s \in H^{\otimes}_B(A_\sigma(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$.

Thm (T.) $\{$dRT cycles$\} = \{$Hodge cycles$\}$ if either

1) $A \otimes_K \bar{K}$ simple, $\dim A$ prime number, $\mathrm{End}_{\bar{K}}(A) \neq \mathbb{Z}$

2)

2) $K = \mathbb{Q}$, for some $\ell$, the $\ell$-adic Tate cycles are fixed by $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. Assume Mumford-Tate conj. holds for $A$. $\ell$-adic Tate cycles $\in H_{\acute{e}t}^{\otimes}(A_{\bar{K}}, \mathbb{Q}_\ell)$ fixed by $\mathrm{Gal}(\bar{\mathbb{Q}}/L)$

Mumford-Tate conj : $H_B^i(-) \otimes \mathbb{Q}_\ell = H_{\acute{e}t}^i(-)$
{$\ell$-adic Tate cycles} = {$\mathbb{Q}_\ell$ - lin combinations of Hodge cycles}

IV Outline of pf

Mumford-Tate gp : $G_{MT} :=$ largest subgp of $GL(H_B^1(-, \mathbb{Q}))$ fixing all Hodge cycles.

$$G_{dR} := \quad " \quad GL(H_{dR}^1(-/K))$$

$G_{dR} \subseteq G_{MT}$

Lemma (Zarkin) $G_1 \subset G_2$ reductive gps in $GL(V)$, $V$ v.s. / field of char 0.

Assume (0) $G_1$ and $G_2$ conn. reductive

(1) $\mathrm{rk}\, G_1 = \mathrm{rk}\, G_2$

(2) $\mathrm{Cent}_{\mathrm{End}(V)} G_1 = \mathrm{Cent}_{\mathrm{End}(V)} G_2$

Then, $G_1 = G_2$.

Fact : $\{dRT \text{ cycles}\} \otimes_{\mathbb{Q}} \mathbb{C} \hookrightarrow H_B^{\otimes}(-, \mathbb{C})$

Hence, enough to show $G_{dR} = G_{MT}$ $(\otimes \mathbb{C})$

For (1), MT holds $\Rightarrow \mathrm{rk}\, G_\ell^\circ = \mathrm{rk}\, G_{MT} \Rightarrow \mathrm{rk}\, G_{dR}^\circ = \mathrm{rk}\, G_{MT}$

For (0), (2) ~~Prop~~ one can construct a Tannakian cat. $\mathcal{M}_{dRT}$ which is mimicing $\mathcal{M}_{AH}$ (by Deligne), with morphism = dRT cycles with $\underline{\mathrm{Aut}^{\otimes}} = G_{dR}$

Rem 1) We need complex conjugation condition, i.e. $dRT \subseteq H_B^{\otimes}(-, \mathbb{R})$

2) This also implies $G_{dR}$ reductive, if you prefer by Tannakian abstract stuff or by a pf of Deligne about some gp being reductive.

In particular,

① $G_{dR}$ is reductive

② $s \in H_{dR}^{\otimes}(\ /L)$ is fixed by $G_{dR}$ $\Leftrightarrow$ $s$ is an $L$-lin combination of $dRT$ cycles

⊛ $\Rightarrow$ $\mathrm{Cent}_{\mathrm{End}(H^1_{dR})} G_{dR}$ = lin. comb. of alg. cycles =

$= \mathrm{Cent}_{\mathrm{End}(H^1_{dR})} G_{MT}$

We want $\mathrm{Cent}_{\mathrm{End}(H^1_{dR})} G_{dR} = \mathrm{Cent}_{\mathrm{End}(H^1_{dR})} G^0_{dR}$

$\left( \xrightarrow{\text{by MT conj.}} G_{dR} \text{ connected} \right).$

## Prime dimensional case

We may assume $A$ is not CM

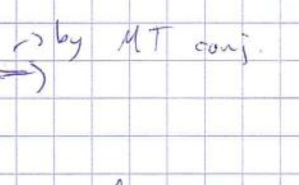
1) $\mathrm{End}^0(A) = F$ tot. real $[F:\mathbb{Q}] = p$

2) $p$ odd $\rightsquigarrow$ $\mathrm{End}^0(A) = F$ CM, $[F:\mathbb{Q}] = 2$

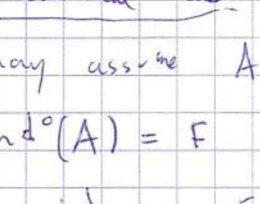
$p = 2$ $\to$ $\mathrm{End}^0(A)$ = quat alg.

Look at rep. of $G_{MT}$ (resp. $G_{dR}$) on $H^1 \otimes \overline{\mathbb{Q}}$

1)




2)



$H^1_{dR} = \underset{\substack{\downarrow \\ \text{irred. of } G_{dR}}}{\bigoplus V_i} = \bigoplus \bigoplus V_{i,j} \ /\overline{K}$

Lem 1. all $V_{i,j}$ are of same dim.

Lem 2. If all $V_{i,j}$ have dim $=1$, i.e. $G^\circ_{dR}$ is a torus,
then $A$ has CM

Lem 3 $V_i \simeq V_j$ as a $G^\circ_{dR}$ rep'n $\Leftrightarrow$ + $G_{dR}$ - rep'n.

Lem 1 ~~Pf~~ Idea: $G^\circ_{dR} \trianglelefteq G_{dR}$
$\leadsto$ $\varphi_v$ fixes same dim. block

$\underline{\quad ^{\circ} _{\circ} \quad}$

Thm (Serre) Extend ~~K~~ K s.t. $G_\ell$ (l-adic cohom) is connected. Then $\exists M$
a set of primes of $K$ of natural density $1_{\text{"}}$ s.t. $\forall v \in M$,
the comm. alg. gp generated by $\mathrm{Frob}_v$ is of maximal rk.
Then, $G$ is connected