Thm. Let $(\mathcal{C}, \otimes)$ be a rigid abelian k-linear tensor category with $k = \mathrm{End}(\mathbb{1})$ a field. Let $\omega: \mathcal{C} \to \mathrm{Vec}_k$ be an exact (faithful) k-linear tensor functor. Then there exists an affine group scheme G/k such that $(\mathcal{C}, \otimes)$ is tensor equivalent to $\mathrm{Rep}_k G$. More precisely, G represents $\underline{\mathrm{Aut}}^{\otimes}(\omega)$.

Such a tuple $(\mathcal{C}, \otimes, \omega)$ is called a neutral tannakian category.

Idea of proof. Let P_X the largest subobject of $\underline{\mathrm{Hom}}(\omega(X), X)$ such that for all $n \in \mathbb{N}$, $Y \subset X^n$, the image of P_X in $\underline{\mathrm{Hom}}(\omega(X^n), X^n)$ lies in $\ker(\underline{\mathrm{Hom}}(\omega(X^n), X^n) \to$ $\to \underline{\mathrm{Hom}}(\omega(Y), X^n/Y))$. Set $A_X = \omega(P_X)$; it is a k-algebra, in fact $A_X = \mathrm{End}^{\otimes}(\omega|_{\langle X \rangle})$. Then take the limit. There is a bialgebra structure. $\square$

Ex.
- $\mathbb{Z}$-graded vector spaces $\leftrightsquigarrow$ representations of $\mathbb{G}_m$.
- X/k smooth variety, $x \in X$, then the category of stratified bundles on X with ω_x the fiber at x is neutral tannakian; here use

$$\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}, \quad \theta \cdot (e \otimes f) = \theta e \otimes f + e \otimes \theta f$$

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}), \quad (\theta \cdot \varphi)(e) = \theta \cdot \varphi(e) - \varphi(\theta e).$$

The Riemann–Hilbert correspondence

Pedro Ángel Castillejo. 23/05/2016

Let $X/\mathbb{C}$ a smooth variety. Recall that we have $\{\mathcal{O}_X\text{-coherent } \mathcal{D}\text{-modules}\} = \{\text{stratified bundles}\} \approx \{\text{vector bundles with flat connection}\}$. If $x \in X$ is a closed point, these categories with the fiber functor F_x are tannakian. Hence they are equivalent to the category of representations of some affine group scheme $\underline{\mathrm{Aut}}^{\otimes} F_x$.

Today we see that, analytically, these categories are equivalent to that of local systems on X^{an}.

euclidean open in $\mathbb{C}^n$

Analytic spaces. An analytic space is a loc ringed space $(X, \mathcal{O}_X)$ that is locally of the form $(U, \mathcal{O}_U)$ where $U = Z(f_1, \dots, f_r) \subset V$, f_i analytic, and $\mathcal{O}_U$ the sheaf of analytic functions. There is analytification $X \rightsquigarrow X^{an}$; locally if $X \subseteq \mathbb{A}^n$ is affine, let $X^{an} = X(\mathbb{C})$ with the subspace top from $\mathbb{C}^n$, and $\mathcal{O}_{X^{an}}$ = analytic functions on X^{an}. In general, glue. One has $\hat{\mathcal{O}}_{X_x} = \hat{\mathcal{O}}_{X^{an},x}$, $\dim \mathcal{O}_X = \dim \mathcal{O}_{X^{an}}$, and $\dim X = \dim X^{an}$. Also for an $\mathcal{O}_X$-module $\mathcal{F}$ there is an $\mathcal{O}_{X^{an}}$-module $\mathcal{F}^{an}$.

Thm (GAGA). Let $X/\mathbb{C}$ be proper.

- $H^i(X, \mathcal{F}) = H^i(X^{an}, \mathcal{F}^{an})$
- $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_{X^{an}}}(\mathcal{F}^{an}, \mathcal{G}^{an})$
- There is an equivalence $\{\text{coherent } \mathcal{O}_X\text{-modules}\} \approx \{\text{coherent } \mathcal{O}_{X^{an}}\text{-modules}\}$.

The Riemann-Hilbert correspondence. Let X be a smooth sep analytic space.

Thm (Riemann-Hilbert). For every local system V on X there is a vector bundle $\mathcal{V} = \mathcal{O}_X \otimes V$ with a unique connection ∇, called canonical, such that for all $v \in \mathcal{V}$ one has $\nabla v = 0$ iff $v \in V$. In particular, for $f \in \mathcal{O}_X$ and $v \in V$ we get $\nabla(fv) = df \otimes v$. The connection ∇ is flat. Moreover, there is an equivalence of tannakian categories

$$\{\text{local systems}\} \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \{\text{flat connections}\}.$$

where $\varphi(V) = \mathcal{O}_X \otimes V$ and $\psi(\mathcal{V}) = \mathcal{V}^{\nabla} := \{v \in \mathcal{V} : \nabla v = 0\}$.

In particular, for $X/\mathbb{C}$ a proper variety, we get with GAGA an equivalence between (algebraic) stratified bundles on X and local systems on X^{an}.

Proof. It is clear that $\mathcal{V}$ is locally free and that $\nabla(fs) = df \otimes v$. Then set $\nabla = d \otimes id : \mathcal{O}_X \otimes V \to \Omega^1_X \otimes V$. We have $\nabla\nabla(fv) = \nabla(df \otimes v) = ddf \otimes v = 0$ so ∇ is a flat connection. Note that for $v \in V$ one has $\nabla(1 \otimes v) = d1 \otimes v = 0$. Conversely, we show that $\Omega^\bullet_X(\mathcal{V})$ is a resolution of V, as in $0 \to V \to \mathcal{V} \to \Omega^1(\mathcal{V}) \to \dots$

- $\operatorname{rk} V = 1$; we may assume $X = D^m$ and $V = \mathbb{C}$, and show that the sequence $0 \to \mathbb{C} \to \Gamma(\mathcal{O}_X) \to \Gamma(\Omega^1_X) \to \dots$ is acyclic. But there is the homotopy operator

$H : \Gamma(\mathcal{O}_X) \to \mathbb{C}$, given by evaluation at 0, and $H : \Gamma(\Omega^1_X) \to \Gamma(\mathcal{O}_X)$ sending $\omega = \sum_{j=1}^{m} \sum_{\alpha \in \mathbb{N}^n} a^j_\alpha x^\alpha dx_j$ to $\sum_{j=1}^{m} \sum_{\alpha \in \mathbb{N}^n} a^j_\alpha x^\alpha \cdot \frac{x_j}{\alpha_j + 1}$, and similar in higher degrees; this works.

- $X = D^m$, V free: follows because exact sequences respect finite direct sums.

For the equivalence we know already that $\psi \circ \varphi \cong \mathrm{id}$ for objects, and for morphisms it is clear. Also $\mathrm{Hom}(V_1, V_2) \to \mathrm{Hom}(\mathcal{V}_1, \mathcal{V}_2)$ is injective, hence surjective by fin dim. It remains to prove that every flat connection comes from a local system.

- $X = D$, $(\mathcal{V}, \nabla)$ free. The difference between ∇ and $d^n : \mathcal{O}_X^n \to \Omega^1_X \otimes \mathcal{O}_X^n$ is an $\mathcal{O}_X$-linear map $\mathcal{O}_X^n \to (\Omega^1_X)^n$. Also Ω^1_X is free of rank 1 so ∇ is given by a 'connection matrix' $\Omega \in \mathrm{Mat}_{n \times n}(\Omega^1_X(X))$, $\Omega = A\,dz$, with $A = (a_{ij})_{i,j} \in \mathrm{Mat}_{n \times n}(\mathcal{O}_X(X))$. Now $\nabla(f) = 0$ iff $df = -Af$, i.e. iff f is a solution of $y' = -Ay$. Hence $\mathcal{V}$ comes from the corresponding sheaf of solutions V, which is locally constant by the Cauchy theorem.
- $X = D$, $(\mathcal{V}, \nabla)$ locally free. Then we can make a presentation $\mathcal{V}_1 \xrightarrow{h} \mathcal{V}_0 \to \mathcal{V}$ with $\mathcal{V}_1, \mathcal{V}_0$ free; and then $(\mathcal{V}, \nabla) = \mathcal{O}_X \otimes (V_0/hV_1)$, where V_0/hV_1 is locally constant.
- In higher dimension, use induction on a relative version of the above.

$\square$

Rmk. What if $U/\mathbb{C}$ is not proper? Choose a smooth compactification $X/\mathbb{C}$ with sncd $D = X \setminus U$, i.e. $D = \bigcup_i D_i$ with D_i smooth crossing transversally. Choose local coordinates $x_1, \dots, x_n$ with $D_i = Z(x_i)$, $i = 1, \dots, r$, so that $\mathrm{Der}_D(X/\mathbb{C})$ is free on $x_i \frac{\partial}{\partial x_i}$, $i = 1, \dots, r$, and $\frac{\partial}{\partial x_j}$, $j = r+1, \dots, n$. Dually we obtain $\Omega^1_X(\log D)$ with basis $\frac{dx_i}{x_i}$, $i = 1, \dots, r$, dx_j, $j = r+1, \dots, n$. We find a correspondence between local systems on U and flat connections with regular singularities on D.