Intersection theory seminar.

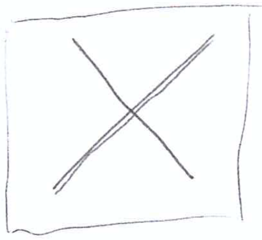
Def. Let X be a scheme. The grop of cycles on X, $Z(X)$ is
$$Z(X) := \bigoplus_{\substack{Y \subset X \\ \text{subvarieties}}} \mathbb{Z} \cdot Y$$

Rem. 1) $Z(X)$ is graded by dimension: $Z(X) = \bigoplus_{k=0}^{\dim X} Z_k(X)$.

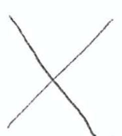
2) Every closed subscheme can be seen as an element of $Z(X)$: is a sum of ~~connected comp~~ irreducible components and the (posible) non-reduced structure is encoded in the coefficients:

Def. $\underset{\text{closed}}{Y \subset X}$ $\rightsquigarrow$ Define $\langle Y \rangle = \sum l_i \cdot Y_i$, where Y_i are the irr. comp. of Y_{red}, and l_i is the multiplicity, i.e., $l_i := \operatorname{length}(\mathcal{O}_{Y, Y_i})$.

e.g.: $X = \mathbb{A}^2_k$
$$\cup$$
$$Y = \operatorname{Spec}\left(k[X_1, X_2] \big/ (X_1 - X_2)^2 (X_1 + X_2) \right)$$



$$\rightsquigarrow$$

$Y_{red} =$  $Y_1 = \operatorname{Spec}\left(k[X_1, X_2]/(X_1 - X_2) \right)$

$Y_2 = \operatorname{Spec}\left(k[X_1, X_2]/(X_1 + X_2) \right)$

$$\rightsquigarrow$$

$$\mathcal{O}_{Y, Y_1} = \left(k[X_1, X_2] \big/ (X_1 - X_2)^2 (X_1 + X_2) \right)_{(X_1 - X_2)}$$

$\rightsquigarrow$ Here the only ~~max~~ ideal that survives the localization is $(X_1 - X_2)$, so we get the chain
$$0 = (X_1 - X_2)^2 \subsetneq (X_1 - X_2) \subsetneq \mathcal{O}_{Y, Y_1}$$

Hence, $l_1 = 2$.

Exercise: $e_2 = 1$.

Def. The l_i's are called multiplicity of Y along the irr. component Y_i, $l_i = \mathrm{mult}_{Y_i}(Y)$. (Fulton: geom. multiplicity).

Def. A cycle $Z = \sum n_i Y_i$ is effective if $n_i \geq 0$.

Def. — " — is a Weil divisor if $Z \in Z_{n-1}(X)$ where X is a pure n-dim. scheme.

Rem. $Z(X) = Z(X_{red})$.

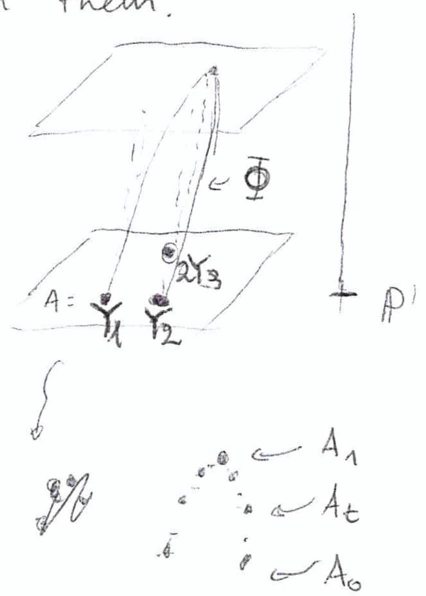
Rational equivalence of cycles

Morally, two cycles $A_0, A_1 \in Z(X)$ are rationally equiv. if there is a (rat.) family of cycles between them.

e.g. $X = \mathbb{P}^2_k$,

$A_0 = Y_1 + Y_2$

$A_1 = 2Y_3$



Def. Two cycles $A_0, A_1 \in Z(X)$ are rationally equivalent if $A_1 - A_2 \in \mathrm{Rat}(X) \subset Z(X)$, where $\mathrm{Rat}(X)$ is generated by differences of the form

$$\langle \Phi \cap (\{t_0\} \times X) \rangle - \langle \Phi \cap (\{t_1\} \times X) \rangle,$$

where $t_0, t_1 \in \mathbb{P}^1$ and $\Phi \subset \mathbb{P}^1 \times X$ is a subvariety not contained in any fiber $\{t\} \times X$.

↳ Why this condition? Else, $\emptyset \sim$ everything $\Rightarrow \mathrm{Rat}(X) = Z(X)$.

Def. Chow ~~group~~: $A(X) := Z(X)/\mathrm{Rat}(X)$

Notation: given a closed subscheme $Y \subset X$, $\langle Y \rangle \in Z(X)$, $[Y] := [\langle Y \rangle]$

Prop. The Chow gp is graded by dimension:
$$A(X) = \bigoplus_k A_k(X), \qquad A_k(X) = Z_k(X)/Rat_k(X).$$

Pf. Given $\Phi \subset \underbrace{\mathbb{P}^1 \times X}_{(t,x)}$ a variety not contained in a fiber over X then $\Phi \cap (\{t_0\} \times X)$ is defined by the vanishing of the non-zero divisor $t - t_0$ (technical remark: we first restrict to $(\mathbb{A}^1 \times X) \cap \Phi$ containing $\Phi \cap (\{t_0\} \times X)$, so we have the coordinate t). of $\Phi \cap \mathbb{A}^1 \times X$.

Recall: Krull's principal ideal theorem: in a noeth. ring, an ideal generated by n elements has codim. $\leq n$.

Hence: $\Phi \cap (\{t_0\} \times X)$ has codimension 1 in Φ.

We do the same for $\Phi \cap (\{t_1\} \times X)$, so we get that two cycles A_0, A_1 rat. equiv. have the same dimension. $\square$

Rem. If X is equidimensional (for example, if X is irred.) we can define the codimension of a subvariety $Y \subset X$, and we can grade $A(X)$ by codimension:
$$A(X) = \bigoplus_{k=0}^{\dim(X)} A_k = \bigoplus_{k=\dim(X)}^{0} A^k$$

Def. The *fundamental class* of X is $[X] \in A(X)$.

Prop. ~~At~~ Let X be a scheme.
$A(X) = A(X_{red})$ $\rightsquigarrow$ This is because $Z(X) = Z(X_{red})$, $Rat(X) = Rat(X_{red})$

b) If X is a variety ~~///~~ of dimension k, then $A_k(X) \simeq \mathbb{Z}$, and is generated by the fundamental class $[X]$.

More generally, if ~~///~~ the irr. components of X are $X_1, \dots, X_m$ the classes $[X_i]$ generate a free ab. subgp of rank m in $A(X)$.

Pf. The $[X_i]$ are generators of $A(X)$.

Now $\mathrm{Rat}(X)$ is generated by varieties Φ in $\mathbb{P}^1 \times X$, each of which is contained in some $\mathbb{P}^1 \times X_i$ (no topological fact! use irreducibility of Φ). $\square$

Ex: $A(\text{zero dim. scheme}) = \bigoplus^{\#\text{comp.}} \mathbb{Z}$.

Rational equivalence via divisors.

We can express the subgp $\mathrm{Rat}(X) \subset Z(X)$ in terms of divisor classes.

$\boxed{1}$ X affine variety: if $f \in \mathcal{O}_X$ is a function on X non-zero, then f defines a subscheme whose irr. components are of codim 1.

Hence is a Weil divisor, we denote it $\mathrm{Div}(f)$.

If $Y \subset X$ closed irr. of codim. 1, we write $\mathrm{ord}_Y(f) :=$ order of the vanishing of f along Y.

Hence,
$$\mathrm{Div}(f) = \sum_{\substack{Y \subset X \\ \text{irr.}}} \mathrm{ord}_Y(f) \cdot \langle Y \rangle$$

Now, if $\alpha = f/g \in K(X)^*$, we define $\mathrm{Div}(\alpha) = \mathrm{Div}(f) - \mathrm{Div}(g)$.

We denote $\mathrm{Div}_0(\alpha) = \mathrm{Div}(f)$, $\mathrm{Div}_\infty(g) = \mathrm{Div}(g)$

$\boxed{2}$ X not affine var.: $K(X) = K(\mathcal{U})$ for any non-empty $\mathcal{U}$, in particular for affines.

Hence, given $\alpha \in K(X)$, we get $\mathrm{Div}(\alpha|_{\mathcal{U}})$. Obviously they agree on overlaps, so they define $\mathrm{Div}(\alpha)$ on X itself.

Fact: $\alpha \longmapsto \mathrm{Div}(\alpha)$ is a hom. from $K(X)^* \longrightarrow \mathrm{Div}(X)$.

Prop. If X is any scheme, $\mathrm{Rat}(X)$ is generated by all divisors of rational functions on all subvarieties of X.
In particular, if X is irr. of dim. n, then $A_{n-1}(X) = A^1(X) = Cl(X)$ $\left(:= \mathrm{Div}(X) / \text{Principal div. } (X) \right)$.

Chow ring:

- We now define the intersection product on $A(X)$.

To see that this is well defined is very hard, we would need half of the seminar to see it. We don't do it, but at least we discuss the definition. First we need some concepts.

Def. Let X be a variety, $A, B \subset X$ subvarieties. A and B intersect ~~generically~~ transversely at $p (\in A \cap B)$ if (i) A, B and X are smooth at p, and (ii) the tangent spaces to A and B at p generate T_pX, i.e. $T_pA + T_pB = T_pX$.

Equivalently, $\operatorname{codim}(T_pA \cap T_pB) = \operatorname{codim} T_pA + \operatorname{codim} T_pB$

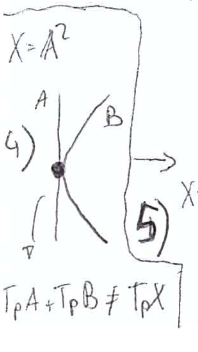
(i.e. $\operatorname{codim}(T_pA + T_pB) = 0$).

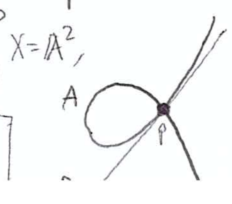
~~Def. $A, B \subset X$ are generi~~

E.g.: 1) Two lines L_1, L_2 in $\mathbb{A}^2_k$ intersect transversely in $p = L_1 \cap L_2$.

2) " " $\mathbb{P}^3_k$ can't intersect transversely, ~~even if~~ (s.t. $L_1 \cap L_2 \neq \emptyset$)

3) Two lines L_1, L_2 in a cone can't intersect transversely, because their intersection will be always in the singularity of the cone.



5) 

$\rightsquigarrow$ Here $T_pA + T_pB = T_pX$, but we don't want here that they intersect transversely. That's why we ask p to be a smooth point of A, B, X.

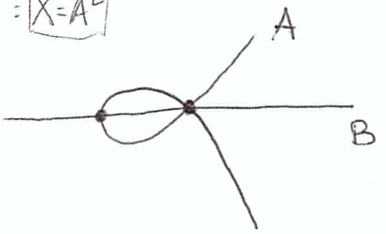
Def. $A, B \subset X$ are <u>generically transverse</u>, or they <u>intersect generically transverse</u>, if they meet transversely at a general pt. of each component C of $A \cap B$. (here we consider $\mathrm{id}: A\cap B \longrightarrow A \cap B$)

More general, two cycles $A, B \in Z(X)$ intersect generically if the
$\underset{\sum n_i A_i}{A}$, $\underset{\sum m_j B_j}{B}$
A_i's and the B_j's intersect generically.

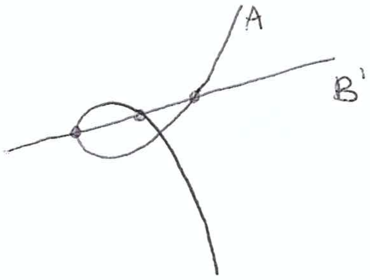
More general: subvarieties $A_i \subset X$ intersect transversely at a (smooth) point $p \in X$ if (i) p is a smooth pt for all the A_i's and X, and (ii) $\operatorname{codim}\left(\bigcap T_p A_i\right) = \sum \operatorname{codim} A_i$.

<u>Exercises</u>: 1) If A and B have complementary dimensions in X, and they are generically transverse, how does $A \cap B$ look like?
2) What ~~also~~ happens if $\operatorname{codim} A + \operatorname{codim} B > \dim X$?

E.g.: $\boxed{X = \mathbb{A}^2}$

1) 

$\rightsquigarrow$ They don't intersect generically transversally.

2) 

$\rightsquigarrow$ They are generically transverse.

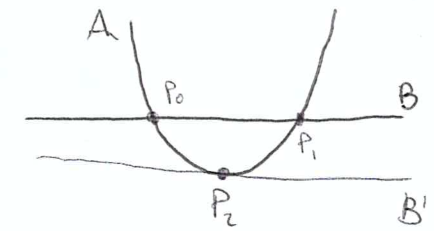
<u>Def</u> (Intersection product).

<u>Thm</u>. Let X be a smooth quasi-proj. variety, then there is a unique product structure on $A(X)$ satisfying the condition

$\boxed{(*) \text{ if two subvarieties } A, B \text{ of } X \text{ are } \underline{\text{gen. transv.}}, \text{ then } [A]\cdot[B] := [A \cap B]}$

This structure makes $A(X) = \bigoplus A^c(X)$ into an associative, commutative ring, graded by codimension, the Chow ring.

Rem. We must assume that A, B intersect transv., otherwise this product is not well defined:



$\rightsquigarrow$ Here $[A \cap B] = [P_0] + [P_1]$,

$[A \cap B'] = [P_2]$

and $[B] = [B']$, while $[P_0] + [P_1] \neq [P_2]$

We will be able to fix this using multiplicities.

The above thm is based in the $\boxed{\text{moving lemma}}$:

Thm. Let X be a smooth quasi-proj. variety.

a) For every $\alpha, \beta \in A(X)$, there are generically cycles $A, B \in Z(X)$ s.t. $[A] = \alpha$, $[B] = \beta$.

b) The class $[A \cap B]$ is independent of the choice of such cycles A and B.

$\hookrightarrow$ This was not proved until 1984, Fulton. $\rightarrow$ Hartshorne ~~says~~ '77 says that this was proved by Chevalley and Roberts before, but Eisenbud and Harris agree with Fulton in saying that these "proofs" were unsatisfactory

Rem. We need the smoothnes assumption.

E.g.: In a cone, two lines intersect always in a singularity, so a) fails.

But we can say something about the singular case, we will do this on Talk 4.

Multiplicities:

Thm. Let $A, B \subset X$ be subvarieties of a smooth variety X s.t. every irr. component C of $A \cap B$ has $\operatorname{codim} C = \operatorname{codim} A + \operatorname{codim} B$.

Then, for each C, there is a positive integer $m_C(A, B)$, called the intersection multiplicity of A and B along C, s.t.

a) $[A][B] = \sum m_C(A, B)[C] \in A(X)$

b) $m_C(A, B) = 1 \iff A$ and B intersect transversely at a general pt of C.

c) $m_C(A, B)$ depends only on the local structure of A and B at a general point of C.

And what is this misterious number? Serre produced a general formula (in '57!) for it. Assuming that A, B are dimensionally transverse (i.e. $\operatorname{codim} C = \operatorname{codim} A + \operatorname{codim} B$ for all ~~connected~~ irr. components C of $A \cap B$), we obtain

$$m_C(A,B) = \sum_{i=0}^{\dim X} (-1)^i \, \mathrm{length}_{\mathcal{O}_{A\cap B, C}} \left(\mathrm{Tor}_i^{\mathcal{O}_{X,C}} (\mathcal{O}_{A,C}, \mathcal{O}_{B,C}) \right)$$

In practice, there are better alternatives to define this, see [Fu, chapter 7].

What will we do in this seminar?

1) Computations of the Chow ring, as well as ~~&~~ important classes.

For this, functorial notions are very important.

2) Grassmannians: we will study a lot grassmannians, specially $G(1,3)$, because knowing $A(G(1,3))$ allows us computing and solving a lot of funny and enumerative problems.
E.g.: given 4 curves $C_1, \dots, C_4 \subset \mathbb{P}^3$ of degrees $d_1, \dots, d_4$, how many lines meet general translates of all 4?

3) Chern classes: these classes of $A(X)$ generalize the correspondance

$$D \longleftrightarrow \mathcal{L}(D)$$

divisors $\qquad$ invertible sheafs

$\cap$

$A^1(X)$

to

$\mathcal{E}$ locally free sheaf of rank r $\longleftrightarrow$ $c_1(\mathcal{E}), \dots, c_r(\mathcal{E})$ Chern classes, $c_i(\mathcal{E}) \in A^i(X)$.

On the 4th talk, we do this for a line bundle, and in the last 5 talks we study more this important notion.

There is a free spot at the end, we accept suggestions! As a possibility, we could explain the Grothendieck-Riemann-Roch thm.

The Chow ring of $\mathbb{A}^n$.

Prop. $A(\mathbb{A}^n) = \mathbb{Z}\cdot[\mathbb{A}^n]$

Pf. Let $Y \subsetneq \mathbb{A}^n$ be a ~~proper~~ closed subvariety. Choose coordinates $z = z_1, \dots, z_n$ so that the origin doesn't lie on Y. Define

$$\Phi^\circ := \{(t, tz) \subset k^* \times \mathbb{A}^n \mid z \in Y\} = V(\{f(z/t) \mid f(z) \text{ vanishes on } Y\})$$

The fiber of W° over $t \in k^*$ is tY, i.e. Y scaled by factor t.

Let $\Phi \subset \mathbb{P}^1 \times \mathbb{A}^n$ be the closure of Φ°. Y irr. $\Rightarrow$

$\Rightarrow \underbrace{(\mathbb{A}^1 \setminus 0) \times Y}_{\cong\, \Phi^\circ}$ irr. $\Rightarrow \Phi$ irr.

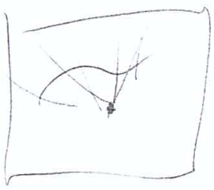
- The fiber of Φ over $t=1$ is just Y.
- Since the origin $\notin Y$, $\exists\, g(z)$ polynomial that vanishes on Y and $g(0) = c \neq 0$.

Now $G(t,z) := g(z/t)$ is defined on $(\mathbb{A}^1 \setminus 0) \times \mathbb{A}^n$ and extends to a regular function on $(\mathbb{P}^1 \setminus 0) \times \mathbb{A}^n$ with constant value c on the fiber $\{\infty\} \times \mathbb{A}^n$.

Hence $\Phi \cap \{\infty\} \times \mathbb{A}^n = \emptyset$, so $\Phi \cap 1 \times \mathbb{A}^n = Y \sim \emptyset \Rightarrow$

$\Rightarrow \langle Y \rangle = 0.$

$\square$

$\boxed{\text{Idea}}$:  at infinity, $\Phi \cap \{\infty\} \times \mathbb{A}^n = \emptyset$!!