Cohomology of $\mathbb{G}_m$ over a curve, 29.06

$\boxed{\text{Let } X \text{ be a proj. smooth connected curve} / k = \bar{k}.}$

Aim of the talk: to compute $H^1(X_{ét}, \mathbb{G}_{m,X})$

Let $\eta \in X$ be its generic point, and $g : \eta \hookrightarrow X$ the inclusion

For a closed pt. $z \in X$, let $i_z : z \hookrightarrow X$ be the inclusion morphism.

Recall from Wouter's talk that we had the exact sequence

$$1 \to \mathcal{O}_X^* \longrightarrow K^* \xrightarrow{\text{val.}} \mathrm{Div}_X \longrightarrow 0$$

where $K = K(X) = K(\eta)$, $\mathrm{Div}_X(\mathcal{U}) = \bigoplus_{\substack{z \subset \mathcal{U} \\ \text{closed of} \\ \text{codim. } 1}} \mathbb{Z} \cdot [z] = \bigoplus_{\substack{z \in \mathcal{U} \\ \text{closed}}} \mathbb{Z} \cdot [z]$.

~~Since~~ This induced, noting that $\mathcal{O}_X^* = \mathbb{G}_{m,X}$, $K^* = g_* \mathbb{G}_{m,\eta}$ and $\mathrm{Div}_X = \bigoplus i_{z,*} \underline{\mathbb{Z}}$

$$\boxed{1 \to \mathbb{G}_{m,X} \longrightarrow g_* \mathbb{G}_{m,\eta} \longrightarrow \bigoplus_{\substack{z \in X \\ \text{closed}}} i_{z,*} \underline{\mathbb{Z}} \to 0}$$

which $\boxed{\text{is an exact sequence on } X_{ét}}$.

Consider the LES:

$$\boxed{\begin{aligned} 0 &\to H^0(X_{ét}, \mathbb{G}_{m,X}) \longrightarrow H^0(X_{ét}, g_* \mathbb{G}_{m,\eta}) \to H^0(X_{ét}, \mathrm{Div}_X) \to \\ &\to H^1(X_{ét}, \mathbb{G}_{m,X}) \longrightarrow H^1(X_{ét}, g_* \mathbb{G}_{m,\eta}) \to H^1(X_{ét}, \mathrm{Div}_X) \to \\ &\longrightarrow H^2(X_{ét}, \mathbb{G}_{m,X}) \to \dots \end{aligned}}$$

$\boxed{\text{Lemma 1: } H^q(X_{ét}, g_* \mathbb{G}_{m,\eta}) = 0 \quad \text{for } q > 0 \qquad (1)}$

$\boxed{\text{Lemma 2: } H^q(X_{ét}, \mathrm{Div}_X) = 0 \quad \text{for } q > 0. \qquad (2)}$

$(1) \Rightarrow \quad H^1(X_{ét}, \mathbb{G}_{m,X}) = \mathrm{coker}\left(\underbrace{H^0(X_{ét}, g_*\mathbb{G}_{m,\eta})}_{} \to H^0(X_{ét}, \mathrm{Div}_X)\right)$

$\wr\!\wr$

$\check{H}^0(X, g_*\mathbb{G}_{m,\eta}) = \cancel{\Gamma(X, \mathcal{O}_X) \simeq k^*} = H^0(X, K^*) \simeq K^* \simeq$ ppal divisors

$\check{\ } \simeq \mathrm{Div}(X)$

Hence, $H^1(X_{ét}, \mathbb{G}_{m,X}) \simeq \mathrm{Div}(X) / \text{ppal divisors} \simeq \mathrm{Pic}(X)$.

We also have

$(1)+(2) \Rightarrow H^i(X_{ét}, \mathbb{G}_{m,X}) = 0$ for $i > 0$.

Summarizing,

$$\boxed{H^q(X_{ét}, \mathbb{G}_{m,X}) = \begin{cases} k^* & \text{for } q = 0 \\ \mathrm{Pic}(X) & \text{for } q = 1 \\ 0 & \text{for } q > 1 \end{cases}}$$

Proof of lemma 1

Step 1: Arithmetic black boxes.

Def. A field K is a C_1 field if every hom. pol. $f(T_1, \dots, T_n) \in K[T_1, \dots, T_n]$ of degree $d < n$ has a non-trivial zero in K^n.

Rem. This fields are called also quasi-algebraically closed: if T_1 occurs in f, set $T_2, \dots, T_n = 1$. We get a polynomial $f(T_1, 1, \dots, 1) \cancel{\tilde{f}} \in K[T_1]$, so K alg. closed $\Rightarrow$ $\cancel{\tilde{f}}$ ($f(T_1, 1, \dots, 1)$) has a zero $\Rightarrow$ f has a non-trivial zero $\Rightarrow$ K is C_1.

Facts

a) Finite fields are C_1.

b) Function fields of dimension $1/_{k=\bar{k}}$ ~~over alg. closed fields~~ are C_1.

c) $K = \mathrm{Frac}(R)$, where R is a henselian dvr R, with alg. closed residue field and $\hat{K}/K$ is separable.

Ex: 1) $K(X) \simeq k(T)$ is C_1

2) Let $R = \mathcal{O}^h_{X,x}$. Then R is a hens. dvr R, its residue field is k and $\underbrace{\hat{K}/K \text{ is separable.}}$?

$\boxed{\text{Black boxes}}$: Let K be C_1, $G := \mathrm{Gal}(K^{sep}/K)$. Then

a) The Brauer gp of K is zero, i.e. $H^2(G, (K^{sep})^\times) = 0$.

b) $H^r(G, M) = 0$ for $r > 1$, M a torsion discrete G-module.

c) $H^r(G, M) = 0$ for $r > 2$, M a discrete G-module.

$G \times M \longrightarrow M$ → discrete top. cts

Step 2. $R^q g_* \mathbb{G}_{m,\eta} = 0$ for $q > 0$.

It is enough to prove this ~~for~~ on stalks.

Let $\mathcal{O}^h_{X,x}$ be the stalk of $\mathcal{O}_X$ at $x \in X$ closed pt.

Let $L = \mathrm{Frac}(\mathcal{O}^h_{X,x})$. Then $L = \mathcal{O}^h_{X,x} \otimes_{\mathcal{O}_X} K(X) \Rightarrow$

$\mathrm{Spec}\, L = \mathrm{Spec}(\mathcal{O}^h_{X,x}) \times_X \eta$

Hence, for $F \in Ab(\eta_{et})$, $(R^q g_* F)_x \overset{\text{by Werter}}{=} H^q(\mathrm{Spec}\,(\underbrace{\mathrm{Frac}\, \mathcal{O}^h_{X,x}}_{L}), F|_{..})$

Hence L is an alg. extension of K, so fact b) + black boxes $\Rightarrow$

$\Rightarrow H^q(\mathrm{Spec}(L), \mathbb{G}_{m,\eta}|_L) = 0$ for $q > 0$.

Step 3 A Leray spectral sequence:

Let $Ab(\eta_{\acute{e}t})$, $Ab(X_{\acute{e}t})$, $\underline{Ab}$, which are ab. categories with enough injectives.

Consider $g_* : Ab(\eta_{\acute{e}t}) \to Ab(X_{\acute{e}t})$, $\Gamma := \Gamma(X_{\acute{e}t}, -) : Ab(X_{\acute{e}t}) \to Ab$, which are left exact and g_* preserves injectives.

Then, " there is a spectral seq. $(R^p\Gamma)(R^q g_*)(F) \Rightarrow R^{p+q}(\Gamma \circ g_*)(F)$ ".

In particular, there exists an exact sequence

$$(*) \quad 0 \to R^1\Gamma(g_* \mathbb{G}_{m,\eta}) \to R^1(\Gamma\circ g_*)(\mathbb{G}_{m,\eta}) \to \Gamma\circ R^1(g_* \mathbb{G}_{m,\eta}) \to$$
$$\to R^2\Gamma(g_* \mathbb{G}_{m,\eta}) \to \cdots$$

Step 4. $H^q(X_{\acute{e}t}, g_* \mathbb{G}_{m,\eta}) = 0$ for $q > 0$.

In $(*)$, note that $R^q(g_* \mathbb{G}_{m,\eta}) = 0$ for $q>0 \Rightarrow$

$$\underbrace{R^q\Gamma(g_* \mathbb{G}_{m,\eta})}_{\substack{\| \\ H^q(X_{\acute{e}t},\, g_* \mathbb{G}_{m,\eta})}} \simeq \underbrace{R^q(\Gamma\circ g_*)(\mathbb{G}_{m,\eta})} \quad \text{for } q>0.$$

$$\begin{tikzcd} \eta \arrow[r, "g"] \arrow[dr] & X \arrow[d, "\pi"] \\ & \mathrm{Spec}(k) \end{tikzcd}$$

$$\begin{tikzcd} Ab(\eta_{\acute{e}t}) \arrow[r, "g_*"] \arrow[drr, bend right, "\Gamma(\eta_{\acute{e}t}, -)"'] & Ab(X_{\acute{e}t}) \arrow[r, "\pi_*"] \arrow[dr, "\Gamma"] & Ab((\mathrm{Spec}\, k)_{\acute{e}t}) \arrow[d, "\Gamma_k"] \\ & & Ab \end{tikzcd}$$

Hence, $H^q(X_{\acute{e}t}, g_* \mathbb{G}_{m,\eta}) = R^q(\Gamma\circ g_*)(\mathbb{G}_{m,\eta}) =$

$$= R^q\Gamma(\eta_{\acute{e}t}, \mathbb{G}_{m,\eta}) = H^q(\eta_{\acute{e}t}, \mathbb{G}_{m,\eta}) =$$

$$= H^q(G(K), K^*) = 0$$

↳ Fact b) + black box

$\square$

Pf. Lemma 2

Recall that for closed immersions $\pi: Z \hookrightarrow \cancel{X}$ we had

$$(\pi_* F)_{\bar{x}} = \begin{cases} F_{\bar{x}} & \text{if } x \in Z \\ 0 & \text{else} \end{cases}$$

Hence, $\iota_Z : Z \longrightarrow X$ being a closed immersion implies that ι_{Z*} is exact, so $H^q(X_{ét}, \iota_{Z*} \mathbb{Z}) = 0$, $q > 0 \Rightarrow$ $H^q(X_{ét}, \mathrm{Div}_X) = 0$ for $q > 0$.

$\square$

Bonus: Poincaré duality

Let X be a Riemann surface (compact orientable surface) of genus g (g holes).

$$H_q(X, \mathbb{Z}) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}^{2g} & q = 1 \\ \mathbb{Z} & q = 2 \end{cases}$$

$$\xrightarrow{P.D.} H^q(X, \mathbb{Z}) = \begin{cases} \mathbb{Z} & q = 0 \\ \mathbb{Z}^{2g} & q = 1 \\ \mathbb{Z} & q = 2 \end{cases} \xrightarrow{\text{with coeff}} H^q(X, \mathbb{Z}/n) = \begin{cases} \mathbb{Z}/n \\ (\mathbb{Z}/n)^{2g} \\ \mathbb{Z}/n \end{cases}$$

Riemann surfaces $\rightsquigarrow$ complex curves

Fact Let X/k be as before. Then, for n prime to char(k),

$$H^q(X_{ét}, \mu_n) = \begin{cases} \mu_n(k) \simeq \mathbb{Z}/n & q = 0 \\ (\mathbb{Z}/n\mathbb{Z})^{2g} & q = 1 \\ \mathbb{Z}/n\mathbb{Z} & q = 2 \\ 0 & \text{else} \end{cases}$$