Category of sheaves on $X_{\acute{e}t}$, 01.06

Recall

Let X be a scheme, consider $\acute{E}t/X$: $\begin{cases} \text{Obj.: étale morph. } U \to X \\ \text{arrows: étale } \varphi : U \to V \text{ over } X \end{cases}$

We endow $\acute{E}t/X$ with the Gr topology given by the coverings $(U_i \to U)$ surjective families of étale morphisms in $\acute{E}t/X$. Denote $X_{\acute{e}t}$

Presheaf: $F : \acute{E}t/X \to Ab$ is a contrav. functor

Sh($X_{\acute{e}t}$) A sheaf in $X_{\acute{e}t}$: F s.t. $F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_U U_j)$ exact $\forall$ $(U_i \to U)$, $U \in ob(\acute{E}t/X)$.

A morphism $F \to G$ is just a natural transformation (i.e. morph. of presheaves and of sheaves are the same thing).

Rem (technical difficulty): $Hom(F, G)$ may not be a set, since $\acute{E}t/X$ is not a small category, i.e. the class ~~of~~ $ob(\acute{E}t/X)$ may not be a set.

Aim of the talk: to see that $Sh(X_{\acute{e}t})$ is an abelian category, and introduce two important exact seq.

Recall. What do we mean by exact seq.?

- For an additive category T (i.e. $Hom_T(F, G)$ has structure of ab. gp, finite direct sums exist), a sequence $0 \to F' \to F \to F''$ is exact in T if $0 \to Hom(T, F') \to Hom(T, F) \to Hom(T, F'')$ is exact. $\forall T \in Ob(T)$

  Analog, $F' \to F \to F'' \to 0$ exact if $0 \to Hom(F'', T) \to Hom(F, T) \to Hom(F', T)$

Clearly $Sh(X_{\acute{e}t})$ and PreSh are additive, ~~as well as PreSh($X_{\acute{e}t}$)~~

Categorical facts:

1. $\cdots \to F_{i-1} \to F_i \to F_{i+1} \to \cdots$ exact $\iff$ seq. of contravariant functors $\cdots \to F_{i-1}(U) \to F_i(U) \to F_{i+1}(U) \to \cdots$ exact
2. $Fun(C^{op}, Ab)$ is abelian. In particular, $PreSh(X_{\acute{e}t})$ is.
3. If a functor $i : C_1 \to C_2$ admits a left adjoint $a : C_2 \to C_1$, (i.e. $Hom_{C_1}(aX_1, X_2) \simeq Hom_{C_2}(X_1, iX_2)$)

then i is left exact.

~~Def~~: Now we are in $Sh(X_{ét})$.

Def. $\alpha : F \to F'$ is loc. surjective if $\forall\, U$ and $s' \in F'(U)$, $\exists$ covering $(U_i \to U)$ s.t.

$$F(U_i) \to F'(U_i) \quad \forall i.$$
$$s \mapsto s'|_{U_i}$$

Rem. Is it surj.? We want to check if $F \xrightarrow{\alpha} F' \to 0 \to 0$ is exact, i.e. if $0 \to Hom(F', T) \to Hom(F, T)$ exact $\forall\, T$.

To show: if $\beta \in Hom(F', T)$ satisfies $\beta \circ \alpha = 0$, then $\beta = 0$.

Let $U \to X$ ét, $s' \in F'(U)$. If $\beta(s') = 0$, we are done.

α loc. surj means $\exists\, (U_i \to U)$ s.t. $s'|_{U_i}$ ~~comes from $F(U_i)$~~ $= \alpha(s_i)$, $s_i \in F(U_i)$, $\forall i$. But $\beta(s')|_{U_i} = \beta(s'|_{U_i}) = \beta \circ \alpha(s_i) = 0$ $\forall$

$\Rightarrow \beta(s') = 0$. □

Indeed, we have more:

Lemma: TFAE

a) $F \xrightarrow{\alpha} F' \to 0$ exact

b) α locally surjective

c) For each geometric point $\bar{x} \to X$, $\alpha_{\bar{x}} : F_{\bar{x}} \to F'_{\bar{x}}$ surj.

Recall: A geom. pt of X over $x \in X$ is $\bar{x} : Spec\, k \to X$ with $k = k^{sep}$ ($= k^{al}$ if k ~~alg~~ perfect) and $im(\bar{x}) = x$.

$$\begin{tikzcd} & U \arrow[d, "?"] \\ Spec\, k \arrow[ur, dashed, "\bar{u}"] \arrow[r, "\bar{x}"'] & X \end{tikzcd}$$

$(U, \bar{u})$ ét. nbd. Stalk: $F_{\bar{x}} = \varinjlim_{(U, \bar{u})} F(U)$

Pf. b) $\Rightarrow$ a) done

a) $\Rightarrow$ c) $\Rightarrow$ b) Milne's notes.

Lemma $F', F, F'' \in ob(Sh(X_{ét}))$. TFAE

a) $0 \to F' \to F \to F''$ exact (in $Sh(X_{ét})$).

b) $0 \to F'(U) \to F(U) \to F''(U)$ exact for all $U \to X$

c) $0 \to F'_{\bar{x}} \to F_{\bar{x}} \to F''_{\bar{x}}$ exact $\forall\, \bar{x} \to X$ geom. point

Rem. $\Gamma(X, -) : Sh(X_{ét}) \to Ab$ defines a left exact functor, so by taking the right derive functor we will be able to define étale cohomology $H^i_{ét}(X, F)$.

Pf. ~~a) clear~~

a) $\Rightarrow$ b) We have the functor $i : Sh(X_{ét}) \to PreSh(X_{ét})$, and later we will see that the sheafification gives a functor $a : PreSh(X_{ét}) \to Sh(X_{ét})$ which is the left adjoint of i. But if i admits a left adjoint $\Rightarrow$ i left exact.

b) $\Rightarrow$ a) Will be clear after we study sheafification.

b) $\Rightarrow$ c) Direct limits preserve exactness of exact seq. of ab. gp.

c) $\Rightarrow$ b) similar to previous lemma.

Here we don't use sheafification!!

$\square$

All together, we have

Prop. Let $0 \to F' \to F \to F'' \to 0$ be a seq. of sheaves of ab. gps on $X_{ét}$. TFAE

a) Seq. is exact

b) $F \to F''$ is loc. surjective

$0 \to F'(U) \to F(U) \to F''(U)$ exact $\forall\, U \to X$.

c) $0 \to F'_{\bar{x}} \to F_{\bar{x}} \to F''_{\bar{x}} \to 0$ exact $\forall$ geom. $\bar{x} \to X$.

Cor. $Sh(X_{ét})$ is abelian.

Pf. Sheafification will given us kernels and cokernels, and $coim(\alpha) \simeq im(\alpha)$ by looking at stalks.

$\square$

Fact. In general, given any ~~category~~ site $\mathcal{C}$, the sheaves on ab. gps $Sh(\mathcal{C})$ is an ab. category.

Kummer sequence

Let ~~A be a k-algebra.~~ k be a field, and consider $\mathbb{G}_{m,k} = \mathrm{Spec}(k[t,t^{-1}])$.

Recall that this induces a sheaf. If A is a k-algebra,

$$\mathbb{G}_{m,k}(A) := \mathrm{Hom}_{k\text{-Sch}}(\mathrm{Spec}\, A, \mathrm{Spec}\, k[t,t^{-1}]) \simeq \mathrm{Hom}_k(k[t,t^{-1}], A) \simeq A^\times$$

Similarly, $\mu_{n,k} = \mathrm{Spec}(k[t]/t^n - 1)$ defines a sheaf and

$$\mathbb{G}_{m,k}(A) = \cdots = \{ n\text{-th roots of } 1 \text{ in } A \}.$$

Let $n \in \mathbb{N}_{>0}$ and consider the ~~Kummer~~ gp hom.

$$0 \to \{t \in A^\times : t^n = 1\} = \mu_n(A) \hookrightarrow A^\times \longrightarrow A^\times$$
$$t \longmapsto t^n$$

Is $A^\times \to A^\times$, $t \mapsto t^n$ surjective? ~~Assume~~ A is ~~henselian~~ local, then ~~we have~~

Let $a \in A$, $T^n - a$ has a root in A? Assume A henselian, then it is enough to see if $T^n - [a]$ has a ~~root in~~ single root in $k = A/m$, because then by Hensel property we lift it to A.

~~If k has char $p > 0$,~~ then

Prop. ~~The seque~~ Let X/k, $\mathrm{char}(k) \nmid n$. Then

$$0 \to \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{t \mapsto t^n} \mathbb{G}_m \longrightarrow 0$$

is exact.

Pf. We have to check surjectivity. By 1^{st} lemma, loc. surj.

Let $U \to X$ étale, $a \in \mathbb{G}_m(U)$. Let $U' = \mathcal{O}_U(U)[T]/(T^n - a)$.

$U' \to U$ is surj., and is étale since $T^n - a$ is separable. For instance,

$\frac{d(T^n - a)}{dT} = nT^{n-1} \neq 0$. Hence $a|_{U'}$ has an n^{th} root in $\mathbb{G}_m(U') \Rightarrow$ loc. surj. $\square$

(char $k \nmid n$!!)

Rem. This sequence is very important. Once we develope a little bit étale cohomology, we obtain the LES

$$0 \to H^0_{ét}(X, \mu_n) \to H^0_{ét}(X, \mathbb{G}_m) \to H^0_{ét}(X, \mathbb{G}_m) \to$$
$$\to H^1_{ét}(X, \mu_n) \to H^1_{ét}(X, \mathbb{G}_m) \to H^1_{ét}(X, \mathbb{G}_m) \to$$
$$\to H^2_{ét}(X, \mu_n) \to H^2_{ét}(X, \mathbb{G}_m) \to \cdots$$

and we get geometric info of X!! Indeed,

$$\mathrm{Pic}(X) \cong H^1_{ét}(X, \mathbb{G}_m)$$
$$\mathrm{Br}(X) \cong H^2_{ét}(X, \mathbb{G}_m)$$

! the image of $H^1_{ét}(X, \mu_n) \to H^1_{ét}(X, \mathbb{G}_m)$ is $\mathrm{Pic}(X)[\ell^n]$

What can we do if $\mathrm{char}\, k \mid n$? $\longrightarrow$ In X_{fl}, is surj. !

Artin-Schreier seq

Recall $\mathbb{G}_a = \mathrm{Spec}\, k[t]$ additive gp. Again, let X/k, $\mathrm{char}\, k = p > 0$

Prop. $0 \to \mathbb{F}_p\ \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{G}_a \xrightarrow{t \mapsto t^p - t} \mathbb{G}_a \to 0$ is exact.

Pf. Is the same, we note here that $\dfrac{d(T^p - T - a)}{dT} = -1 \neq 0$.

Rem. There exists an exact seq. that unifies both.

Sheafification

Fix a site $\mathcal{C}$, and consider a presheaf of sets $\mathcal{P}$.

Def. Let $\mathcal{P} \to a\mathcal{P}$ be a hom. (of presheaves) from $\mathcal{P}$ to a sheaf $a\mathcal{P}$. We say that $a\mathcal{P}$ is the sheaf associated with $\mathcal{P}$ (or sheafification) if

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[dr] & a\mathcal{P} \arrow[d, dashed, "\exists !"] \\ & \mathcal{F} \end{tikzcd} \qquad \forall \mathcal{F} \text{ sheaf}$$

i.e. $\mathrm{Hom}(\mathcal{P}, \mathcal{F}) \simeq \mathrm{Hom}(a\mathcal{P}, \mathcal{F})$

Prop. Every presheaf $\mathcal{P}$ on $X_{\acute{e}t}$ has an associated sheaf $sh: \mathcal{P} \to a\mathcal{P}$. The ~~functor~~ map sh induces isomorphisms $\mathcal{P}_{\bar{x}} \to (a\mathcal{P})_{\bar{x}}$ on the stalks. The functor $a: \mathrm{PreSh}(X_{\acute{e}t}) \to \mathrm{Sh}(X_{\acute{e}t})$ is exact.

Sketch of pf

Step 1: Construction of $\mathcal{P}^*$: for each $x \in X$, choose a geom. point $\bar{x} \to X$ over x. $\mathcal{P}_{\bar{x}}$ is an ab. gp.

Recall (skyscraper ~~presheaf~~): If Λ is an ab gp, define $\Lambda^{\bar{x}}$ as

$$\Lambda^{\bar{x}}(\mathcal{U}) = \bigoplus_{\mathrm{Hom}_X(\bar{x}, \mathcal{U})} \Lambda \qquad \left(\text{if } X/k = \bar{k},\ \Lambda^{\bar{x}}(\mathcal{U}) = \Lambda^{x}(\mathcal{U}) = \bigoplus_{u \in \varphi^{-1}(x)} \Lambda, \quad \varphi: \mathcal{U} \to X\right)$$

Then $\mathcal{P}^* = \prod (\mathcal{P}_{\bar{x}})^{\bar{x}}$ is a sheaf.

~~Note that~~

Step 2: Lemma: Let $sh: \mathcal{P} \to \mathcal{F}$ be a hom. from a presheaf $\mathcal{P}$ to a sheaf $\mathcal{F}$. If sh satisfies

a) the only sections of $\mathcal{P}$ to have the same image in $\mathcal{F}(\mathcal{U})$ are those that are locally equal

b) sh is loc. surjective

Then $(\mathcal{F}, sh)$ is the sheafification of $\mathcal{P}$.

Note that $\mathcal{P}^*$ satisfies a), but it may be too big.

Step 3 Lemma. Let $\mathcal{P}$ be a subpresheaf of a sheaf $\mathcal{F}$. For each $\mathcal{U} \to X$, let $\mathcal{P}'(\mathcal{U})$ be the set of $s \in \mathcal{F}(\mathcal{U})$ that are locally in $\mathcal{P}$, i.e. $\exists$ covering $(\mathcal{U}_i \to \mathcal{U})$ s.t. $s|_{\mathcal{U}_i} \in \mathcal{P}(\mathcal{U}_i)\ \forall i$.

Then, $\mathcal{P}'$ is a subsheaf of $\mathcal{F}$, and $\mathcal{P} \to \mathcal{P}'$ is locally surj.

Pf. Easier to think that to write.

Def. $\mathcal{P}'$ is the subsheaf of $\mathcal{F}$ generated by $\mathcal{P}$.

~~Hence, we~~

~~Hence, we~~ Step 4. Let $a\mathcal{P}$ be the subsheaf of $\mathcal{P}^*$ generated by $\mathcal{P}$. Then $a\mathcal{P}$ satisfies a) and b).

Step 5 If $sh: \mathcal{P} \to \mathcal{F}$ satisfies a) and b), then $sh_{\bar{x}}: \mathcal{P}_{\bar{x}} \to \mathcal{F}_{\bar{x}}$ is an isom. $\forall$ geom. points.

Step 6. $0 \to \mathcal{P}' \to \mathcal{P} \to \mathcal{P}'' \to 0$ exact $\Rightarrow$

$\Rightarrow$ $0 \to \mathcal{P}'_{\bar{x}} \to \mathcal{P}_{\bar{x}} \to \mathcal{P}''_{\bar{x}} \to 0$ exact $\Rightarrow$

$\Rightarrow$ $0 \to (a\mathcal{P}')_{\bar{x}} \to (a\mathcal{P})_{\bar{x}} \to (a\mathcal{P}'')_{\bar{x}} \to 0$ exact

$\Rightarrow$ $0 \to a\mathcal{P}' \to a\mathcal{P} \to a\mathcal{P}'' \to 0$ exact

$\square$

Remark We can define a sheafification as follows for an ~~general~~ arbitrary site $\mathcal{C}$.

$\mathcal{F}$ presheaf of ab. gps on $\mathcal{C}$

$$\rightsquigarrow \mathcal{F}^+(\mathcal{U}) := \varinjlim \operatorname{eq}\left(\prod_i \mathcal{F}(\mathcal{U}_i) \rightrightarrows \prod_{i,j} \mathcal{F}(\mathcal{U}_i \times_{\mathcal{U}} \mathcal{U}_j)\right)$$

Fact 1: $\mathcal{F}^+$ is separated presheaf, i.e. in the seq.

$$\mathcal{F}^+(\mathcal{U}) \xrightarrow{\rho} \prod \mathcal{F}^+(\mathcal{U}_i) \rightrightarrows \prod \mathcal{F}^+(\mathcal{U}_i \times_{\mathcal{U}} \mathcal{U}_j)$$

we have $\operatorname{im}(\rho) \subseteq \operatorname{eq}$.

Fact 2: If $\mathcal{F}$ is separated, $\mathcal{F}^+$ is a sheaf.

Hence, $(\mathcal{F}^+)^+$ is a sheaf.