Homotopy, deformation retraction and contractibility. 04.02.2015

Notation: map = continuous map, $I = [0,1]$

Def (Homotopy): let $f_0, f_1 : X \to Y$ continuous maps. They are <u>homotopic</u> if there exists a <u>homotopy</u> ~~F~~ F between them, i.e. a continuous map $F : X \times I \to Y$ s.t. $f_0 = F(-,0)$ and $f_1 = F(-,1)$. We write $f_0 \simeq f_1$.

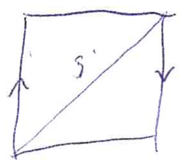
Ex (0) Let $X = [0,2]^2$, $Y = [0,2]$, $f_0 : X \to [0,2] : (x_1,x_2) \mapsto x_1$ and $f_1 : X \to [0,2] : (x_1, x_2) \mapsto x_2$. Then $f_0 \simeq f_1$.
Indeed, $F : [0,2]^2 \times [0,1] \longrightarrow [0,2] : ((x_1,x_2), t) \mapsto t x_2 + (1-t) x_1$ is the homotopy between f_0 and f_1.

Def (Retraction). A <u>retraction</u> of X onto A is a map $r : X \to X$ s.t. $r(X) = A$ and $r|_A = id_A$.

Def. (Deformation retraction). A <u>~~(strong)~~ deformation retraction</u> of X onto a subspace $A \subset X$ is a homotopy $F: X \times I \to X$ connecting id_X and ~~a~~ a retraction $r : X \to A$.
We say that it is <u>strong</u> if $f_t|_A = id_A$ $\forall t \in I$.

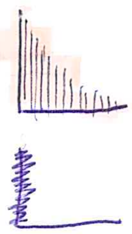
Ex. (1) Möbius band and S^1.

Let $M = [0,1]^2 / (0,x) \sim (1, 1-x)$ be the Möbius band, and let $S^1 := \{[(x_1,x_2)] \in M \mid x_1 = x_2\}$ be a circle.



Then ~~there~~ $\exists$ strong deformation retraction of M onto S^1.
Indeed, let $F: M \times [0,1] \longrightarrow M : ((x_1,x_2), t) \mapsto (x_1, t x_1 + (1-t) x_2)$
$f_0 = id_M$, $f_1 = r : M \twoheadrightarrow S^1$ retraction onto S^1.

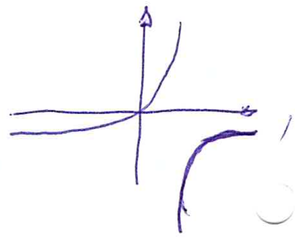
(2) Consider the space $X := [0,1]\times\{0\} \cup \bigcup_{q\in\mathbb{Q}\cap[0,1]} \{q\}\times[0,1-q]$,

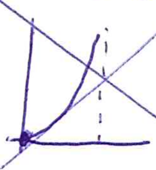
and the subspace ~~A~~ $X \supset A := [0,1]\times\{0\}$.  X A

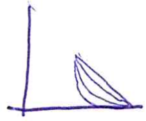
We construct a deformation retraction of X onto A.

Define first, for $t\in[0,1)$, ~~$\alpha_t(x)$~~:

$$\alpha_t : [1,2] \to [0,1] : x \mapsto \alpha_t(x) = (2-x)\,\frac{1}{x^{t/(1-t)}}$$

Don't panic: from high-school, $\frac{t}{1-t}$ is something like 

so in $[0,1)$ we have  with

$\frac{t}{1-t} \to \infty$ as $t\to 1$. 

Then, $\alpha_t(x) \xrightarrow[t\to 1]{} 0$ $\forall x>1$, and $\alpha_t(1) = 1$ $\forall t\in[0,1)$.

Define $\alpha_1 : [1,2] \to [0,1]$: as $\alpha_1(x) = \begin{cases} 1 & \text{if } x=1 \\ 0 & \text{else} \end{cases}$

Then $A : [1,2]\times I \to [0,1] : A(x,t) = \alpha_t(x)$ is a homotopy connecting $\alpha_0 : x \mapsto (2-x)$ and α_1.

Now we can define the deformation retraction:

$$F : X\times I \to X : ((x_1,x_2),t) \mapsto \begin{cases} (x_1,x_2) & \text{if } x_2 \le \alpha_t(1+x_1) \\ (x_1, \alpha_t(1+x_1)) & \text{else} \end{cases}$$

~~PR~~

Def. (Homotopy equivalence) A homotopy equiv between X and Y is a map $f: X\to Y$ s.t. there exists $g: Y\to X$ with $g\circ f \simeq id_X$, $f\circ g \simeq id_Y$. We denote $X \simeq Y$.

Rem (1) If X deformation retracts onto A, then $r\circ\iota = id_A$ and $\iota\circ r \simeq id_X$. Hence, $X \simeq A$.

Thm. If $X \simeq Y \Rightarrow H_n(X) \cong H_n(Y)$ $\forall n$.

Here we prove a smaller result:

Prop. If A is a retract of X, then the maps $H_n(A) \to H_n(X)$ are injective.

Marcel's remark!

Pf. $H_n(-)$ is a functor $\forall n$. Hence, $id_{H_n(A)} = H_n(id_A) =$

$$= H_n(r\circ\iota) = H_n(r)\circ H_n(\iota) \Rightarrow H_n(\iota): H_n(A) \to H_n(X) \text{ inj.} \quad \square$$

Def (Contractibility). Let X be a space, $x_0 \in X$. We say that X is contractible if $X \simeq \{x_0\}$. Equivalently, id_X is homotopic to $X \to x_0 \hookrightarrow X$

Rem (2) X contractible $\Rightarrow$ X path-connected. Indeed, let $F: X\times I \to X$ be the homotopy connecting id_X and the constant map x_0. Let $\tilde{x} \in X$. Then $F(\tilde{x}, -): I \to X : t \mapsto f_t(\tilde{x})$ is a path from $\tilde{x}$ to x_0.

~~Ex~~ (3) X path-connected $\not\Rightarrow$ X contractible! S^1.

Ex. (3) A segment is trivialy contractible. Therefore, the space X from (2) is contractible, since  $\simeq$ L $\simeq$ $\cdot$ (point).

- ~~If X has a (strong) deformation retract onto a point, by definition is contractible (see Rem. (1)). We ask ourselves if the converse is~~

$$X \text{ (strong) def. retr. onto a point} \underset{\Leftarrow ?}{\Rightarrow} \text{contractible}$$

- The answer is no. Before we see a ~~exam~~ counterexample, we prove the following thm.

Theorem → Before the proof, write application!

Thm. If a space X has a strong def. retract onto $x_0 \in X$, ~~.~~ then $\forall\, U \underset{op.}{\subseteq} X$, $x_0 \in U$, $\exists\, V \underset{op.}{\subseteq} U$, $x_0 \in V$ s.t. the inclusion $\iota : V \hookrightarrow U$ is homotopic to a constant map ~~x~~

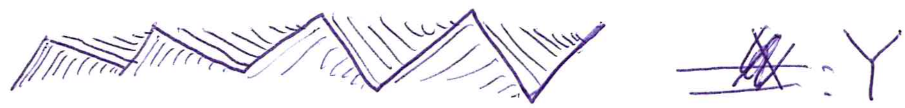
$x_0 : V \longrightarrow U : V \longmapsto x_0$.

Ex. (4). Thm $\Rightarrow$ $X =$  doesn't have a strong def. retract onto (x_1, x_2) if $x_2 \neq 0$, since ~~every~~ $\exists\, \varepsilon_0 > 0$ s.t. $\forall\, B_\varepsilon(x_0)$, $0 < \varepsilon < \varepsilon_0$, $B_\varepsilon(x_0) \cap X$ is not path-connected (indeed, ∞-many connected components!!).

Pf [

Ex (5) Contractible $\not\Rightarrow$ strong deformation retract onto a point.

Consider

$Y :=$ 

~~As~~ Similarly as in example (2),

$Y \simeq$ 

and a line is obviously contractible. Therefore Y is contractible.

But! If it had a strong def. retract onto $y_0 \in Y$, then $\forall\, U \ni y_0$, $\exists\, V \subset U$ s.t. V is contractible.

But given $y_0 \in Y$, $\exists\, \varepsilon_0 > 0$ s.t. $B_\varepsilon(y_0)$ ~~has~~ is not path-connected $\forall\ 0 < \varepsilon < \varepsilon_0$ $\lightning$.