Fundamental Groups and Covering Spaces

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Why invariants?

Definition

Let X and Y be topological spaces. We say that a continuous map $f : X \to Y$ is a homeomorphism if it has a continuous inverse, i.e. if there is a continuous map $g : Y \to X$ with $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.
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Our failure to write down a homeomorphism proves nothing. So we need some invariant which helps us distinguishing spaces.

The Fundamental Group

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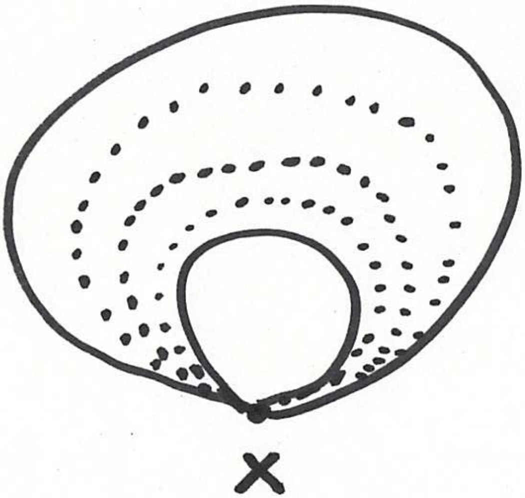
The set $L(X)$ is already an invariant of X - but a really bad one. So we have to be more clever.


x

The Fundamental Group

Definition

Let $f, g : [0,1] \to X$ be two loops in X. We say that f and g are homotopic if there is a continuous map $H : [0,1] \times [0,1] \to X$ with $H(-,0) = f$, $H(-,1) = g$ and $H(0,t) = H(1,t) = x$ for all t, and we say that H is a homotopy from f to g.

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Currently, $\pi_1(X,x)$ is only a set. But it can be turned into a group.

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$$(f \star g)(t) = \begin{cases} f(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

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It is not hard to check that the homotopy class of $f \star g$ in $\pi_1(X, x)$ does only depend on the homotopy classes of f and g in $\pi_1(X, x)$. Hence we get a well-defined multiplication on $\pi_1(X, x)$.

The Fundamental Group

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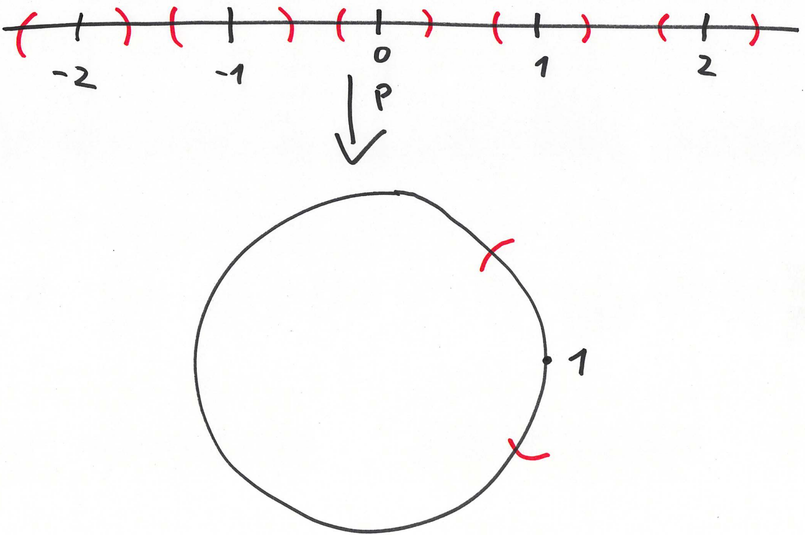
3. The inverse of f is the loop f' obtained from f by running in the other direction: $f'(t) = f(1 - t)$.
4. Hence $\pi_1(X, x)$ is a group, called the fundamental group of X.

Covering maps

Basic example of a covering map:

$$p : \mathbb{R} \to S^1, t \mapsto e^{2\pi it}$$

Covering maps



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Definition

Assume X is path-connected. Let $p : \tilde{X} \to X$ be a continuous map with $\tilde{X}$ also path-connected. We say that p is a covering map if the following holds:
Each point $x \in X$ has a neighborhood U such that there is a homeomorphism $p^{-1}(U) \cong p^{-1}(x) \times U$ under which the projection $p : p^{-1}(U) \to U$ corresponds to the projection onto the second factor $p^{-1}(x) \times U \to U$.

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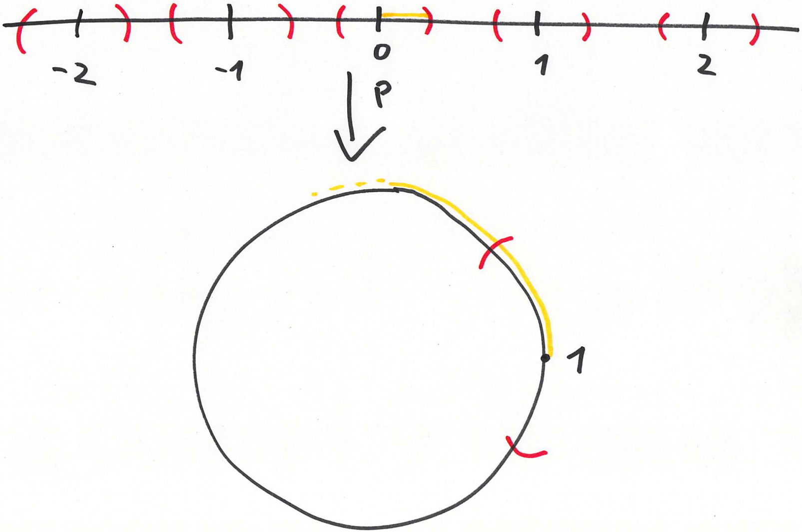
If in addition $\pi_1(\tilde{X}) = 0$, we say that $\tilde{X}$ is the universal cover of X.



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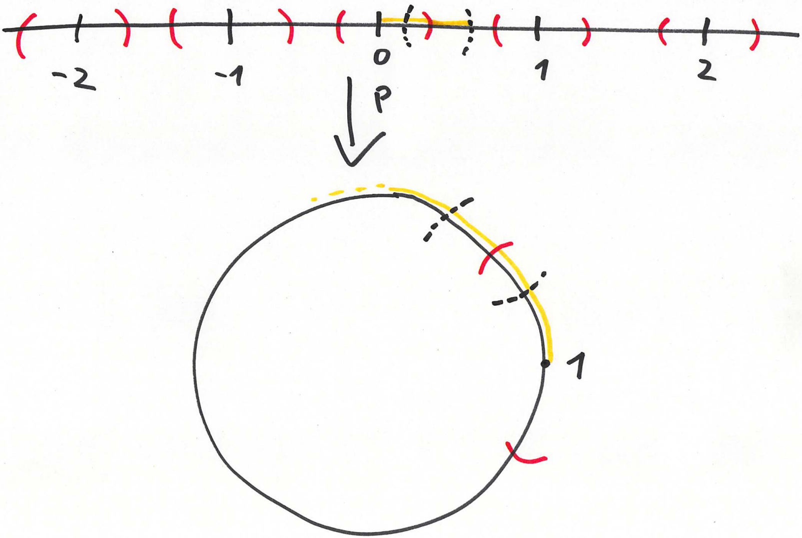
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Lifting properties

Proposition

Let $p : \tilde{X} \to X$ be a covering map. Let $f : [0,1] \to X$ be a loop at x. Pick $\tilde{x} \in \tilde{X}$ with $p(\tilde{x}) = x$. Then there is a unique path $\tilde{f} : [0,1] \to \tilde{X}$ with $\tilde{f}(0) = \tilde{x}$ lifting f, i.e. such that $p \circ \tilde{f} = f$.

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If H is a homotopy from f to g, there is a unique homotopy $\tilde{H} : [0,1] \times [0,1] \to \tilde{X}$ lifting H, i.e. such that $p \circ \tilde{H} = H$. This homotopy fixes the endpoints of $\tilde{f}$ and $\tilde{g}$. Conversely, each homotopy of paths from $\tilde{f}$ to another path $\tilde{h}$, fixing the endpoints, gives rise to a homotopy of loops from $f = p \circ \tilde{f}$ to $h = p \circ \tilde{h}$.

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So the homotopy class of the loop f is uniquely determined by the homotopy class of the path $\tilde{f}$, where we demand that homotopies fix the endpoints of our paths.

Reading off $\pi_1(X)$

If in addition $\tilde{X}$ is the universal cover, i.e. $\pi_1(\tilde{X}) = 0$, one can show that there is a unique homotopy class of paths connecting two given points of $\tilde{X}$.

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It follows that the elements of $\pi_1(X)$ are in $1-1$ correspondence with $p^{-1}(x)$: Fix $\tilde{x} \in p^{-1}(x)$. For each $y \in p^{-1}(x)$, there is a path $\tilde{f}$, unique up to homotopy, from $\tilde{x}$ to y. Then $f = p \circ \tilde{f}$ is a loop.

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But what about the group structure?

Deck transformations

Definition

A deck transformation of $\tilde{X}$ is a homeomorphism $\phi : X \to X$ such that $p \circ \phi = p : \tilde{X} \to X$

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For each $y \in p^{-1}(x)$, there is a unique deck transformation $\phi : \tilde{X} \to \tilde{X}$ such that $\phi(\tilde{x}) = y$.

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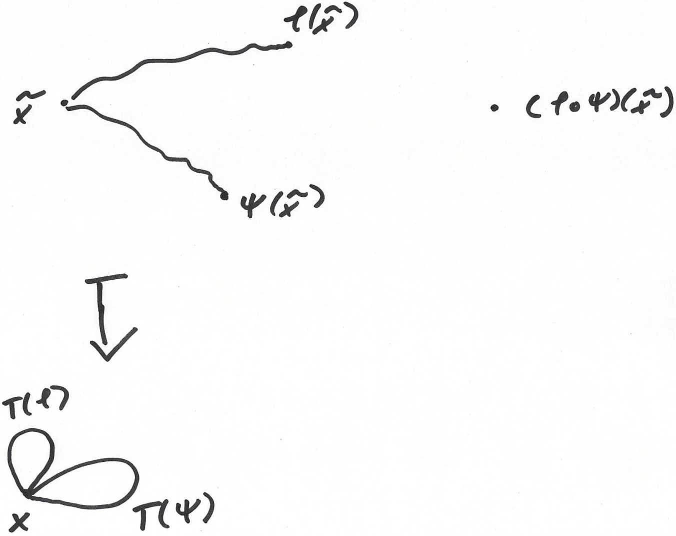
Each deck transformation ϕ determines a loop $T(\phi)$: Pick a path $\tilde{f}$ from $\tilde{x}$ to $\phi(\tilde{x})$ and form the loop $f = p \circ \tilde{f}$.
This gives rise to a group homomorphism $T : G \to \pi_1(X, x)$.



φ(x̃)
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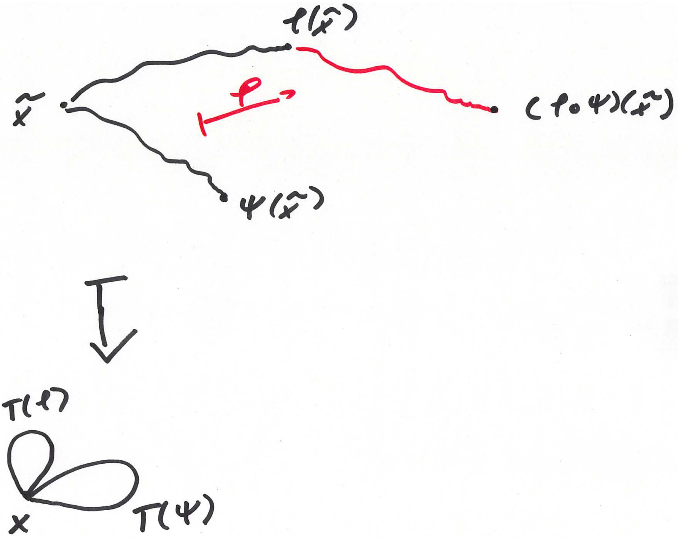
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Hence the fundamental group can be read off from the group of deck transformations, which is often easy to determine.