Topology 3

Problem Set 2 H. Reich/F. Lenhardt
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Exercise 1
Show that there is no space X such that $X \times X \cong \mathbb{R}^3$.
Hint: Consider $H^*(X \times X, X \times X - \{(p,p)\})$ for some $p \in X$ and try to compute it in two different ways.

Exercise 2
We consider $\mathbb{R}\mathrm{P}^k$ as a subspace of $\mathbb{R}\mathrm{P}^n$. Show that for $n \geq 2$ and $k < n$ $\mathbb{R}\mathrm{P}^k$ is not a retract of $\mathbb{R}\mathrm{P}^n$.

Exercise 3
Let A and B be graded commutative R-algebras. Show that the graded tensor product $A \widehat{\otimes}_R B$ has the universal property of a coproduct in the category of graded commutative R-algebras.

Exercise 4
Let $A_1, \ldots, A_n \subset \mathbb{R}^n$ be compact measurable sets.

(i) For $p \in S^{n-1} \subset \mathbb{R}^n$, let $E_p \subset \mathbb{R}^n$ be the hyperplane orthogonal to p. Show that there is a closed interval $[a_p, b_p] \subset \mathbb{R}$ (where $a_p = b_p$ is allowed) such that for $\lambda \in [a_p, b_p]$, the affine hyperplane $\lambda p + E_p$ cuts A_n into two parts of the same size.

(ii) Let $\lambda_p = \frac{a_p + b_p}{2}$. Convince yourself that λ_p depends continuously from p. No written proof is required.

(iii) We define the positive side of a hyperplane $\lambda p + E_p$ to be the side to which p is pointing. For $p \in S^{n-1}$ and $1 \leq i \leq n-1$, let $V_i(p)$ be the volume of the part of A_i which is on the positive side of $\lambda_p p + E_p$. Use the function $f\colon S^{n-1} \to \mathbb{R}^{n-1}$ given by

$$f(p) = (V_1(p), V_2(p), \ldots, V_{n-1}(p))$$

to conclude that there is an affine hyperplane cutting all A_i into two parts of the same size.

(iv) Given a ham sandwich, consisting of two layers of bread and a layer of ham, we can make a single cut with a knife which halves both layers of bread and the ham.

Remark: This is not an exercise in measure theory; you may use the notion of volume in a naive way.