Topology 3

Problem Set 1 — H. Reich/F. Lenhardt

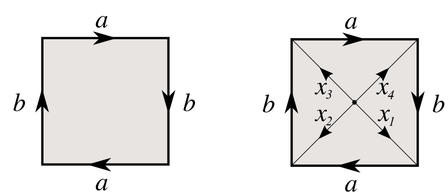
SS 2013 — Due:

Exercise 1

Determine the cup product structure on $H^*(\coprod_{i=1}^n S^1;\mathbb{Z})$.

Exercise 2

When one identifies the sides of a square as indicated in the picture below,one obtains $\mathbb{R}\mathrm{P}^2$. A Δ-complex structure on $\mathbb{R}\mathrm{P}^2$ is indicated by the right-hand picture below.



Show:

(i) $[a+b]$ is a generator for $H_1^\Delta(X;\mathbb{F}_2)$.

(ii) If $\gamma \in C^1_\Delta(X;\mathbb{F}_2)$ is given by

$$\gamma(a) = \gamma(x_1) = \gamma(x_4) = 1$$

and $\gamma(x) = 0$ for all other 1-Simplices, then $[\gamma]$ is a generator of $H^1_\Delta(X;\mathbb{F}_2)$.

(iii) $[\gamma] \cup [\gamma]$ is a generator of $H^2_\Delta(X;\mathbb{F}_2) \cong \mathbb{F}_2$.

Exercise 3

Assume we had not already defined the cup product, but only the cross product for singular cohomology with coefficients in a ring R. Assume we already knew that the cross product has the following properties:

(i) R-bilinearity

(ii) Associativity

(iii) graded commutativty

(iv) Naturality

Define a Cup-product with the help of the cross product and show that it has the properties (i) to (iv).

Hint: Consider the diagonal map $\Delta\colon X \to X \times X$.

Please turn the page!

Exercise 4
We will show that the ring structure on $H^*(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2)$ is given by

$$H^*(\mathbb{R}\mathrm{P}^n; \mathbb{F}_2) \cong \mathbb{F}_2[X]/(X^{n+1}) \quad \text{for } n \geq 1.$$

Use this to compute the ring structure on $H^*(\mathbb{R}\mathrm{P}^4; \mathbb{Z})$.