Topology 3

Problem Set 5 — H. Reich/F. Lenhardt

SS 2013 — Due: 03.07.

Exercise 1

Let $f\colon X \to Y$ be a pointed map.

(i) Define a (non-trivial) "coaction map"

$$\mu\colon \operatorname{hocof}(f) \to \Sigma X \vee \operatorname{hocof}(f)$$

such that for all pointed spaces Z, the map

$$[\Sigma X, Z] \times [\operatorname{hocof}(f), Z] \cong [\Sigma X \vee \operatorname{hocof}(f), Z] \to [\operatorname{hocof}(f), Z]$$

defines an action of the group $[\Sigma X, Z]$ on the set $[\operatorname{hocof}(f), Z]$.

(ii) Show that $a, b \in [\operatorname{hocof}(f), Z]$ are mapped to the same element in $[Y, Z]$ if and only if they are in the same orbit of this action, i.e. if and only if there is $g \in [\Sigma X, Z]$ with $ga = b$.

Exercise 2

Let X be a finite connected pointed CW-complex. Suppose Y is a pointed CW-complex with $\pi_i(Y)$ finite for $0 \leq i \leq \dim(X)$. Show that $[X, Y]$ is finite.
Hint: One can use Exercise 1.

Exercise 3

Let $\mathcal{C}, \mathcal{D}$ be categories and let $L\colon \mathcal{C} \to \mathcal{D}$ respectively $L'\colon \mathcal{C} \to \mathcal{D}$ be left adjoint to $R\colon \mathcal{D} \to \mathcal{C}$ respectively $R'\colon \mathcal{D} \to \mathcal{C}$. Let $\eta\colon L \to L'$ be a natural transformation. Show that there is a natural transformation $\tau\colon R' \to R$ such that the diagram

$$\begin{array}{ccc}
\operatorname{Hom}_{\mathcal{C}}(L'X, Y) & \xrightarrow{\cong} & \operatorname{Hom}_{\mathcal{D}}(X, R'Y) \\
\downarrow{\scriptstyle \eta_X^*} & & \downarrow{\scriptstyle \tau_Y^*} \\
\operatorname{Hom}_{\mathcal{C}}(LX, Y) & \xrightarrow{\cong} & \operatorname{Hom}_{\mathcal{D}}(X, RY)
\end{array}$$

commutes, where the horizontal arrows are the adjunction isomorphisms.

Exercise 4

Given a pointed homology theory $\tilde{h}_*(-)$ on the category of CW-complexes and a pair (X, A), we define $h_*(X, A) = \tilde{h}_*((X \cup CA)_+)$, where Z_+ for a space Z is Z with a disjoint basepoint added. Show that h_* defines an unpointed homology theory.