Freie Universität Berlin

Homotopy Theory

Problem Set 1 — H. Reich/F. Lenhardt
SS 2013 — Due: 05.06.

Exercise 1
Let $i\colon A \hookrightarrow X$ be a cofibration with A contractible. Show that the canonical map $p\colon X \to X/A$ is a homotopy equivalence.

Exercise 2
Let $E = (E, e_0)$ and $B = (B, b_0)$ be pointed spaces and let $f\colon E \to B$ be a pointed map. Let

$$PB = \{\alpha \in B^I \mid \alpha(0) = b_0\} \text{ and } N(f) = \{(\alpha, e) \in B^I \times E \mid \alpha(1) = f(e)\}.$$

Show that:

(i) The map

$$\begin{array}{rccc} \mathrm{ev}_1\colon PB & \to & B \\ \alpha & \mapsto & \alpha(1) \end{array}$$

is a fibration

(ii) The map

$$\begin{array}{rccc} \pi(f)\colon N(f) & \to & B \\ (\alpha, e) & \mapsto & \alpha(0) \end{array}$$

is a fibration.

Exercise 3
For $n \geq 2$, consider the fibration sequences

$$S^1 \hookrightarrow S^{2n+1} \xrightarrow{p_n} \mathbb{C}P^n \qquad \text{and}$$

$$S^3 \hookrightarrow S^{4n+3} \xrightarrow{q_n} \mathbb{H}P^n.$$

(i) Show that the inclusion $S^1 \hookrightarrow S^{2n+1}$ is nullhomotopic.

(ii) Show that the inclusion $S^3 \hookrightarrow S^{4n+3}$ is nullhomotopic.

(iii) Analyze the associated long exact sequences of homotopy groups.

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Exercise 4
A diagram $X_1 \xrightarrow{i_1} X_1 \sqcup X_2 \xleftarrow{i_2} X_2$ in the category $\mathcal{C}$ is a sum diagram if for all objects Y, the map

$$\begin{array}{ccc} \mathrm{mor}_{\mathcal{C}}\left(X \sqcup X', Y\right) & \longrightarrow & \mathrm{mor}_{\mathcal{C}}\left(X, Y\right) \times \mathrm{mor}_{\mathcal{C}}\left(X', Y\right) \\ f & \mapsto & (i_1^*(f), i_2^*(f)) \end{array}$$

is a bijection.
Let $\mathcal{C} \xrightarrow{L} \mathcal{D}$ and $\mathcal{D} \xrightarrow{R} \mathcal{C}$ be functors with a natural bijection

$$\tau_{X,Y} \colon \mathrm{mor}_{\mathcal{D}}\left(LX, Y\right) \longrightarrow \mathrm{mor}_{\mathcal{C}}\left(X, RY\right).$$

(i) Give a precise formulation what naturality means.

(ii) Show that $L(-)$ maps sum diagrams in $\mathcal{C}$ to sum diagrams in $\mathcal{D}$.

(iii) Prove a statement about a connection between $X_1 \vee X_2$ and the functor $- \wedge Y$.