# Topology 2

Problem Set 14 H. Reich/F. Lenhardt
WS 2012/13 Due: 06.02.2013

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**Exercise 1**
Let $X$ be a $CW$-complex with

$$H_n(X,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n=0 \\ \mathbb{Q} & \text{if } n=1 \\ 0 & \text{else} \end{cases}$$

Show that for $R = \mathbb{F}_p$ and $R = \mathbb{Q}$ the Euler characteristic

$$\chi_R(X) = \sum_{i\in\mathbb{Z}} (-1)^i \dim_R H_n(X,R)$$

is well defined and that $\chi_{\mathbb{F}_p}(X) \neq \chi_{\mathbb{Q}}(X)$. Does this contradict any theorems from the lecture?

**Exercise 2**
Let $X$ be a connected $CW$-complex with fundamental group $\pi_1(X,x_0) = G$. We assume that the universal cover $\widetilde{X}$ of $X$ is contractible. Show that:

(i) The singular chain complex $C^{sing}(\widetilde{X})$ yields a projective resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$; compare exercise 3 of problem set 12.

(ii) We have $\mathrm{Tor}_n^{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z}) \cong H_n(X;\mathbb{Z})$

**Exercise 3**
Any path $\sigma\colon ([0,1],\{0,1\}) \to (X,x_0)$ can be regarded as a singular 1-simplex. Show that in this way we obtain a well-defined, surjective map

$$h\colon \pi_1(X,x_0) \to H_1(X;\mathbb{Z})$$

*Remark:* One can show that $h$ defines an isomorphism $h\colon \pi_1(X,x_0)_{ab} \to H_1(X;\mathbb{Z})$.

**Exercise 4**
Let $X,Y$ be spaces. What can we say about the ring structure in $H^*(X\vee Y,\mathbb{Z})$?