Topology 2

Problem Set 12 — H. Reich/F. Lenhardt
WS 2012/13 — Due: 23.01.2013

Exercise 1
Show that $\mathbb{Q}$ is not projective as a $\mathbb{Z}$-module.

Hint: Consider the map $p\colon \mathbb{Z}[\mathbb{N}] \to \mathbb{Q}$, $\sum_{n\in\mathbb{N}} \lambda_n [n] \mapsto \sum_{n\in\mathbb{N}} \lambda_n \frac{1}{n}$.

Exercise 2
Prove or disprove:

(i) For each family of $\mathbb{Z}$-modules M_{ij}, $(i,j) \in I \times J$, the natural map

$$\bigoplus_{i\in I} \prod_{j\in J} M_{ij} \longrightarrow \prod_{j\in J} \bigoplus_{i\in I} M_{ij}$$

is an isomorphism.

(ii) For each $\mathbb{Z}$-module M the functor

$$\operatorname{Hom}_{\mathbb{Z}}(M,-)\colon \mathbb{Z}\text{–}\mathcal{MOD} \longrightarrow \mathbb{Z}\text{–}\mathcal{MOD}$$

is additive.

(iii) For each $\mathbb{Z}$-module M the functor

$$\operatorname{Hom}_{\mathbb{Z}}(M,-)\colon \mathbb{Z}\text{–}\mathcal{MOD} \longrightarrow \mathbb{Z}\text{–}\mathcal{MOD}$$

strongly additive, i.e., commutes with infinite direct sums.

Exercise 3
Let $C_2 = \langle t \mid t^2 \rangle$ be the group with two elements. Let $\mathbb{Z}[C_2]$ be its group ring, i.e., the multiplication is given by the bilinear extension of the multiplication in the group. The map $t \mapsto 1$ extends to a ring homomorphism $\epsilon : \mathbb{Z}[C_2] \to \mathbb{Z}$ and in particular $\mathbb{Z}$ becomes a $\mathbb{Z}[C_2]$-module.

(i) Construct a projective resolution $P_\bullet \to \mathbb{Z}$ of $\mathbb{Z}[C_2]$-modules such that $P_i = \mathbb{Z}[C_2]$ for all $i \geq 0$.
Hint: $(1-t)(1+t) = 0$.

(ii) Compute $\operatorname{Tor}_n^{\mathbb{Z}[C_2]}(\mathbb{Z},\mathbb{Z})$ and $\operatorname{Tor}_n^{\mathbb{Z}[C_2]}(\mathbb{Z},\mathbb{F}_2)$ for all $n \geq 0$. Compare this with $H_n(\mathbb{R}P^\infty;\mathbb{Z})$ and $H_n(\mathbb{R}P^\infty;\mathbb{F}_2)$.

Exercise 4
For a finitely generated abelian group A let

$$\mathrm{t}(A) = \{x \in A \mid \exists\, n \in \mathbb{Z}, n \neq 0 \text{ with } nx = 0\}.$$

Show that

$$\mathrm{t}(A) \neq 0 \;\Rightarrow\; \operatorname{Ext}^1_{\mathbb{Z}}(A,\mathbb{Z}) \neq 0.$$

Try to find a proof which also works when we do not assume that A is finitely generated.