Freie Universität Berlin

Topology 2

Problem Set 11 — H. Reich/F. Lenhardt

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Exercise 1

Let R be a commutative ring. A subset $S \subset R$ is multiplicatively closed if $1 \in S$ and if $a, b \in S$ implies $ab \in S$. For an R-module M we set

$$S^{-1}M = (S \times M)/\sim .$$

where the equivalence relation $\sim$ is given by

$$(s, m) \sim (s', m') \Leftrightarrow \exists\, t \in S \text{ such that } ts'm = tsm'.$$

The equivalence class of (s, m) is denoted $\frac{m}{s}$. Show that this defines a functor

$$S^{-1}\colon R\text{--}\mathcal{MOD} \longrightarrow S^{-1}R\text{--}\mathcal{MOD}$$

Discuss inhowfar this functor is an exact functor.

Exercise 2

Let

$$0 \longrightarrow C \xrightarrow{f} D \xrightarrow{g} E \longrightarrow 0 \qquad (*)$$

be a short exact sequence of chain complexes of R-modules. Prove or disprove:

(i) If there is a chain map $s\colon E \to D$ with $g \circ s = \mathrm{id}_E$, then for all $n \in \mathbb{Z}$ the boundary map $\partial_{n+1}\colon H_{n+1}(E) \to H_n(C)$ in the long exact sequence associated to $(*)$ is zero.

(ii) If the sequence $(*)$ is degreewise split exact, then for all $n \in \mathbb{Z}$ the boundary map ∂_{n+1} is zero.

Exercise 3

Compute

$$\mathrm{Hom}_{\mathbb{Z}}\left((\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Q}) \otimes (\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Q}), \mathbb{Z} \oplus \mathbb{Q} \oplus \prod_{m=1}^{\infty} \mathbb{Z}/m\mathbb{Z}\right)$$

for $n \geq 1$.

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Exercise 4
Let

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0 \qquad (*)$$

be a short exact sequence of $\mathbb{Z}$-modules. Let $C(X)$ be the singular chain complex of a space X.

(i) Show that there is a short exact sequences of chain complexes

$$0 \longrightarrow C(X) \otimes_{\mathbb{Z}} L \longrightarrow C(X) \otimes_{\mathbb{Z}} M \longrightarrow C(X) \otimes_{\mathbb{Z}} N \longrightarrow 0$$

(ii) Show that there is a long exact sequence

$$\ldots \longrightarrow H_n(X; L) \longrightarrow H_n(X; M) \longrightarrow H_n(X; N) \longrightarrow H_{n-1}(X; L) \longrightarrow \ldots$$

(iii) Find an example of a short exact sequence $(*)$ as above and a space X such that the boundary map in the long exact sequence is non-zero at some point.