Topology 2

Problem Set 9 — H. Reich/F. Lenhardt

WS 2012/13 — Due: 19.12.2012

Exercise 1

Construct a topological space X such that

$$H_0(X) = \mathbb{Z}, \quad H_1(X) = \mathbb{Z}/2, \quad H_2(X) = \mathbb{Z} \text{ und } H_3(X) = \mathbb{Z} \oplus \mathbb{Z}/2.$$

Exercise 2

For $A \in GL_{n+1}(\mathbb{R})$ define $f_A \colon S^n \to S^n$ via $x \mapsto \frac{Ax}{\|Ax\|}$. Show that

$$\deg(f_A) = \frac{\det(A)}{|\det(A)|}.$$

Exercise 3

Show that if the closed orientable surface Σ_g of genus g is an n-sheeted covering space of Σ_h, then $g = n(h-1)+1$.

Exercise 4

Let X be a finite graph, i.e. a finite one-dimensional CW-complex, which is connected and does not have vertices of valence 1.

(i) Find an explicit description of the elements of $H_1(X)$.

(ii) Now assume that $H_1(X) \neq \mathbb{Z}$ and $H_1(X) \neq 0$, and assume that X has no vertices of valence 2. Let $f \colon X \to X$ be a homeomorphism such that $H_1(f)$ is the identity. Show that f preserves edges and orientations of edges, i.e. f maps each edge to itself in an orientation-preserving way.
Hint: Show that the map f is automatically cellular. A cellular map induces a self-map of the cellular chain complex in the evident way. You may use that the map on homology induced by this map of chain complexes is $H_*(f)$.

(iii) Show that if additionally $f^k = \mathrm{id}_X$ for some k, we must have $f = \mathrm{id}_X$.

(iv) Show that for a finite subgroup G of the group of homeomorphisms of X, the map

$$G \to \mathrm{Aut}(H_1(X))$$
$$g \mapsto H_1(g) \colon H_1(X) \to H_1(X)$$

is injective.