Topology 2

Problem Set 4 — H. Reich/F. Lenhardt
WS 2012/13 — Due: 14.11.2012s

Exercise 1
For a commutative ring R, let $\mathcal{R}-\mathcal{MOD}$ be the category of R-modules and R-module homomorphisms. A functor $F\colon \mathcal{R}-\mathcal{MOD} \to \mathcal{R}-\mathcal{MOD}$ is *exact* if it preserves long exact sequences, i.e. if for each long exact sequence

$$\dots \xrightarrow{a_{i+2}} A_{i+1} \xrightarrow{a_{i+1}} A_i \xrightarrow{a_i} A_{i-1} \xrightarrow{a_{i-1}} \dots$$

also its image under F

$$\dots \xrightarrow{F(a_{i+2})} F(A_{i+1}) \xrightarrow{F(a_{i+1})} F(A_i) \xrightarrow{F(a_i)} F(A_{i-1}) \xrightarrow{F(a_{i-1})} \dots$$

is a long exact sequence. Show that a functor F is exact if and only if it preserves short exact sequences, i.e. if for each short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

also its image under F

$$0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \longrightarrow 0$$

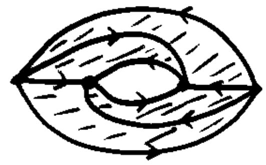
is a short exact sequence.

Exercise 2
Let $X = \mathbb{R}^n_{\geq 0} = \{(x_1, \dots, x_n) \mid x_1 \geq 0\} \subset \mathbb{R}^n$. Let $x, y \in X$ be two points together with a neighborhood U of x and a neighborhood V of y such that there is a homeomorphism of pairs $(U, y) \cong (V, x)$, i.e. a homeomorphism $U \cong V$ mapping x to y. Show that either both points lie on the boundary of X or both lie in the interior of X.

Exercise 3
Consider $\mathbb{Q}$ as a subspace of $\mathbb{R}$. Show that $H_1(\mathbb{R}, \mathbb{Q})$ is an infinitely generated free abelian group.

Please turn the page!

Exercise 4
Consider the Δ-complex X which is indicated in the picture:



Consider the chain complex

$$\Delta_2(X) \overset{\partial_2}{\to} \Delta_1(X) \overset{\partial_1}{\to} \Delta_0(X).$$

(i) Find two essentially different cycles $z_1, z_2 \in \mathrm{Ker}(\partial_1) \subset \Delta_1(X)$ whose homology classes generate $H_1^{\Delta}(X) \cong \mathbb{Z}$.

(ii) Find y in $\Delta_2(X)$ with $z_1 - z_2 = \partial_2(y)$.

(iii) Consider the Δ-subcomplex $A = \partial X$, i.e. the union of the closed 1-simplices which are contained in precisely one 2-simplex.Show that $H_2^{\Delta}(X, A) \cong \mathbb{Z}$ and find an explicit cycle

$$z \in \ker\left(\Delta_2(X)/\Delta_2(A) \overset{\partial_2}{\to} \Delta_1(X)/\Delta_1(A)\right),$$

whose homology class generates $H_2^{\Delta}(X, A) \cong \mathbb{Z}$.