Exercises for

Topology 2

Problem set 2 H. Reich/F. Lenhardt
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Exercise 1

Let X, Y_1 and Y_2 be topological spaces. Let $p_i \colon Y_1 \times Y_2 \to Y_i$, $i = 1, 2$, be the projections. We obtain a map

$$\psi\colon\ \operatorname{map}(X, Y_1 \times Y_2) \to \operatorname{map}(X, Y_1) \times \operatorname{map}(X, Y_2)$$
$$f \mapsto (p_1 \circ f, p_2 \circ f)$$

where all mapping spaces carry the compact-open topology.

(i) Show that ψ is continuous and bijective.

(ii) Show that if, in addition, X is locally compact, ψ is a homeomorphism.

Exercise 2

We consider the ascending union $\mathbb{R}^0 \subset \mathbb{R}^1 \subset \mathbb{R}^2 \subset \dots$ with respect to the inclusions $\mathbb{R}^n \to \mathbb{R}^{n+1}$, $\quad (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0)$. On the space

$$\mathbb{R}^\infty = \bigcup_{n=0}^{\infty} \mathbb{R}^n$$

we consider two topologies: On the one hand, we have the colimit topology, i.e. $U \subset \mathbb{R}^\infty$ is open if and only if $U \cap \mathbb{R}^n$ is open in $\mathbb{R}^n$ for all $n \in \mathbb{N}_0$. On the other hand, $\mathbb{R}^\infty$ can be considered as a subspace of the Hilbert space

$$l^2(\mathbb{N}) = \{(x_1, x_2, \dots) \mid \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$$

and hence as a metric space via the metric induced from the norm

$$||(x_1, x_2, \dots)||_2 = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}$$

Show that:

(i) The subspace topology on $\mathbb{R}^n \subset \mathbb{R}^\infty$ does not depend on which of the two topologies we put on $\mathbb{R}^\infty$.

(ii) The map
$$\begin{array}{rcl} f\colon \mathbb{R}^\infty & \to & \mathbb{R} \\ (x_1, x_2, \dots) & \mapsto & \sum_{i=0}^{\infty} x_i \end{array}$$
is continuous with respect to the colimit topology, but not with respect to the metric topology.

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Exercise 3
Let n_0, $n_2 \geq 1$, $n_1 \geq 0$ be natural numbers with $n_0 - n_1 + n_2 = 2$. Find a CW-structure on S^2 with n_0 0-cells, n_1 1-cells and n_2 2-cells.

Exercise 4
Let V be a euclidean finite-dimensional vector space and let S^V be its one-point compactification. Show that for euclidean, finite-dimensional vector spaces V and W, there is a homeomorphism

$$S^{V \oplus W} \cong S^V \wedge S^W$$