Lecture 3

3.1 Examples (concerning $\otimes$ and $*$)

R in this section will always be a PID

1a As an easy exercise you are supposed to determine for any R-module M the modules

$$R/_{r\cdot R} \otimes M \quad \text{and} \quad R/_{r\cdot R} * M$$

R, as usual, a PID, $r \in R \setminus \{0\}$.

1.b An R-module M is called <u>torsion free</u> if for any $m \in M \setminus \{0\}$, $r \in R \setminus \{0\}$ we have $r \cdot m \neq 0$.

<u>Remark</u>: 1a will show that $R/_{r\cdot R} * M \neq 0$ if and only if $\exists\, m \in M \setminus \{0\}$ with $r m = 0$.

1.c <u>Proposition</u>: If M is torsion free then for any R-module N we have $M * N = 0$

<u>Proof</u>. Let $0 \to F_1 \xrightarrow{\partial_1} F_0 \to 0$. $p: F_0 \to M$ be a free resol. of M, choose bases B_0, B_1 of F_0 and F_1 and consider an element $f: B_1 \to N \in F_1 \otimes N$ in the kernel of $\partial_1 \otimes N$. Let $\{a_1, \dots, a_r\} = f^{-1}(N \setminus \{0\})$. Then for every $1 \le i \le r$ there exists a finite $C_i < B_0$ and $r_{bi} \in R$ such that

$$a_i = \sum_{b \in C_i} r_{bi}\, b$$

Let $C = \bigcup_i C_i \subset B_0$; C is finite ; let $F_0^C \subset F_0$

be the free submodule with basis C, let $M^C := p(F_0^C)$

and $F_1^C = \ker p^C$, $p^C := p|_{F_0^C}$.

By construction, $a_1, \dots, a_r \in F_1^C = F_1 \cap F_0^C$

(We consider F_1 as a submodule of F_0; $F_1 = \ker p$, ∂_1 is the inclusion)

Every element x of $F_1^C \subset F_1$ can uniquely be written

as $x' + x''$ with x' a linear combination of $a_1, \dots, a_r$

and x'' a linear combination of elements in $B_1 - \{a_1, \dots, a_r\}$

It follows that $\{a_1, \dots, a_r\}$ can be extended to a basis

of F_1^C so that f can be considered an element of

$F_1^C \otimes N$ which is non-zero if the original f was non-zero,

and f is still in the kernel of $\bar{\partial}_1 \otimes N$, where

$\bar{\partial}_1 : F_1^C \hookrightarrow F_0^C$.

Now, M^C is finitely generated and torsion free.

Then M^C is free, and therefore $M^C * N = 0$. So

f is in fact 0. $\square$

3.2 The elements $m\otimes n \in M\otimes N$ for

$m\in M$, $n\in N$.

2a. Choose a free resolution $0 \to F_1 \xrightarrow{\partial_1} F_0 \to 0$, $p: F_0 \to M$,

pick a basis B_0 for F_0 and $x \in F_0$ s.t. $p(x) = m$.

Let $x = \sum_{b\in B_0} r_b b$ and define $f \in F_0 \otimes N$ by

$$f(b) = r_b \cdot n \quad , \quad b \in B_0 .$$ Then we define

$$m\otimes n = [f] \in H_0(F\otimes N).$$

2b. This is well-defined.

(i) given the resolution F, p it does not depend on the choice of x. If x' is another choice, then $x'-x \in F_1$; if B_1 is any basis of F_1

$$x'-x = \sum_{a\in B_1} \lambda_a a \quad ; \text{ pick } g: B_1 \longrightarrow N$$

s.t. $g(a) = \lambda_a \cdot n$. Let $a = \sum_{b\in B_0} \lambda_{ba} b$

Then $(\partial_1 \otimes N)(g)(b) = \sum_{a\in B_1} \lambda_{ba} \cdot \lambda_a \cdot n$

$$x' = x + \sum_{a\in B_1} \lambda_a a = \sum_{b\in B_0} \Big(r_b + \sum_{a\in B_1} \lambda_a \lambda_{ba}\Big) b$$

Thus, using x', $m \otimes n = [f']$ with

$$f'(b) = \Big(r_b + \sum_{a \in B_1} \lambda_a \lambda_{ba}\Big) n \quad , \quad \text{and}$$

$$(f' - f)(b) = (\partial_1 \otimes N)(g); \text{ thus } [f'] = [f].$$

(ii) $m \otimes n$ does not depend, up to the natural isomorphisms, on the choice of basis for F_0

(iii) $m \otimes n$ does not depend on the choice of resolution, again up to the corresponding natural isomorphism, coming from the chain map $\alpha : F \to F'$ which makes

$$\begin{array}{ccc} F_0 & \xrightarrow{\alpha_0} & F_0' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ M & = & M \end{array}$$

commute. This easily follows from (i) and (ii).

2c If $E \subset M$ and $D \subset N$ are generating sets of M and N as R-modules then

$$m \otimes n, \quad m \in E, \; n \in D, \quad \text{generate } M \otimes N.$$

In fact, the following is easy to show

$$r(m \otimes n) = (rm) \otimes n = m \otimes rn$$

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$$

$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$$

The equalities are established by making the appropriate choices in 2a. For example to show the last equation: pick $x \in F_0$ for the definition of $m \otimes n_1$ <u>and</u> $m \otimes n_2$, <u>and</u> $m \otimes (n_1 + n_2)$.

Then $f_i : B_0 \longrightarrow N$ is $b \longmapsto r_b \cdot n_i$, $i = 1, 2$

$$\text{And } \underset{\substack{\| \\ m \otimes (n_1+n_2)}}{[f_1 + f_2]} = \underset{\substack{\| \\ m\otimes n_1}}{[f_1]} + \underset{\substack{\| \\ m \otimes n_2}}{[f_2]}$$

□

2d: If $\alpha : M \to M'$ and $\beta : N \to N'$ are R-homom. then $\alpha \otimes \beta : M \otimes N \longrightarrow M' \otimes N'$ maps $m \otimes n$ to $\alpha(m) \otimes \beta(n)$.

3.3 Exactness properties. As with Hom and Ext, applying $\otimes$ to a short exact sequence leads to a six-term exact sequence. i.e.

If $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$

is a natural short exact sequence of R-modules, then there is a natural R-morphism $\partial : C * N \longrightarrow A \otimes N$ s.t.

$$0 \to A * N \xrightarrow{\alpha * N} B * N \xrightarrow{\beta * N} C * N \xrightarrow{\partial} A \otimes N \xrightarrow{\alpha \otimes N} B \otimes N \xrightarrow{\beta \otimes N} C \otimes N \to 0$$

is exact (natural with respect to short ex. sequences and R-modules N)

<u>Proof</u>. We have seen that there is a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \ker p \arrow[r, "i_1"] \arrow[d] & \ker(p,q') \arrow[r, "\pi_2"] \arrow[d] & \ker q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_0 \arrow[r, "i_1"] \arrow[d, "p"] & F_0 \oplus G_0 \arrow[r, "\pi_2"] \arrow[d, "(p,q')"] & G_0 \arrow[r] \arrow[d, "q"] & 0 \\
0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d] & B \arrow[r, "\beta"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with F_0, G_0 free, all rows and all columns exact, and $q' : G_0 \longrightarrow B$ is any homom. such that $\beta \circ q' = q$ (q' exists since G_0 is free and β is surjective), $i_1 : F_0 \longrightarrow F_0 \oplus G_0$ is the obvious inclusion, and $\pi_2 : F_0 \oplus G_0 \longrightarrow G_0$ the obvious projection.

Since the top and middle row split, tensoring with $\otimes N$ gives again splitting short sequences, so that setting

$$F := 0 \to \ker p \to F_0 \to 0$$

$$G := 0 \to \ker q \to G_0 \to 0$$

$$E := 0 \to \ker(p,q') \to F_0 \oplus G_0 \to 0$$

gives a short exact sequence of R-chain complexes

$$0 \to F \otimes N \xrightarrow{i_1 \otimes N} E \otimes N \xrightarrow{\pi_2 \otimes N} G \otimes N \to 0$$

The corresponding long exact homology sequence gives the desired 6-term sequence.

3.4 Universal coefficient theorem for homology.

4a Let C and D be R-chain complexes then we define $C\otimes D$ by

$$(C\otimes D)_n := \bigoplus_i C_i \otimes D_{n-i}$$

and $\partial^{C\otimes D}$ by $\partial^C \otimes D + C\otimes \partial^D$

$= (\partial^C \otimes id_D + id_C \otimes \partial^D)$ where

$(\partial^C \otimes D + C\otimes \partial^D)_n$ is defined on the summand

$C_i \otimes D_{n-i}$ on a generator $c_i \otimes d_{n-i}$ by

$$\underbrace{\partial_i^C(c_i)\otimes d_{n-i}}_{\in C_{i-1}\otimes D_{n-i}} + (-1)^i \underbrace{c_i \otimes \partial^D_{n-i}(d_{n-i})}_{\in C_i \otimes D_{n-i-1}}$$

One checks that $C\otimes D$ is a chain complex.

4b We consider first the case that D is a module N (i.e. $D_0 = N$, $D_i = 0$, $i\neq 0$). Then $C\otimes N$ is in degree n $C_n \otimes N$ and $(\partial^{C\otimes N})_n = \partial_n^C \otimes N$.
$H_n(C\otimes N)$ is called the n-th homology group of C with coefficients in N.

4c Universal coeff. thm., algebraic version

If C is a free R-chain complex, and N an R-module then there exists a natural (in C and N) short exact sequence (for every n)

$$0 \to H_n(C) \otimes N \xrightarrow{\alpha} H_n(C \otimes N) \xrightarrow{\beta} H_{n-1}(C) * N \to 0$$

The sequence splits, but not naturally. Furthermore, α maps $[z] \otimes n$ to $[z \otimes n]$, for any cycle $z \in Z_n(C)$ and $n \in N$.

<u>Proof</u>. Similar to the coeff. thm for cohomology.

Consider (as before)

$$0 \to Z \xrightarrow{i} C \xrightarrow{\partial} sB \to 0$$

This is a short exact sequence of free chain complexes, where $\partial^Z = 0$ and $\partial^{sB} = 0$. Tensoring with N gives a short exact sequence of chain complexes

$$0 \to Z \otimes N \xrightarrow{i \otimes N} C \otimes N \xrightarrow{\partial \otimes N} sB \otimes N \to 0$$

The boundary maps in $Z \otimes N$ and $sB \otimes N$ are still 0. So we get a long exact sequence

$$\cdots \to (sB \otimes N)_{n+1} \to (Z \otimes N)_n \xrightarrow{i \otimes N} H_n(C \otimes N) \to (sB \otimes N)_n \to (Z \otimes N)_{n-1} \to$$

and one checks that $(sB \otimes N)_{n+1} = B_n \otimes N \to Z_n \otimes N = (Z \otimes N)_n$ is simply $j_n \otimes N$, where $j_n : B_n \to Z_n$ is the inclusion

So we get a short exact sequence

$$0 \longrightarrow \operatorname{coker}(j_n \otimes N) \longrightarrow H_n(C \otimes N) \longrightarrow \ker(j_{n-1} \otimes N) \longrightarrow 0$$

and since $0 \to B_n \xrightarrow{j_n} Z_n \xrightarrow{[\ ]} H_n(C) \to 0$ determines a free resolution of $H_n(C)$ we obtain the desired sequence. Since $(i_n \otimes N): Z_n \otimes N \longrightarrow C \otimes N$ maps $z_n \otimes n$ to $z_n \otimes n$ considered as element of $C \otimes N$, α maps $[z_n] \otimes n$ to $[z_n \otimes n]$. To show that the sequence splits, choose some splitting

$$\sigma_{n-1}: B_{n-1} \longrightarrow C_n \quad \text{of} \quad \partial_n : C_n \longrightarrow B_{n-1} = (sB)_n$$

If $f \in B_{n-1} \otimes N$ is in the kernel of $j_{n-1} \otimes N$

$f: E_{n-1} \longrightarrow N$, E_{n-1} a basis of B_{n-1}.

Then $(\partial_n \otimes N)(\sigma_{n-1} \otimes N)(f) = f$; but f is 0 in $Z_{n-1} \otimes N$, therefore 0 in $C_{n-1} \otimes N$. So $(\sigma_{n-1} \otimes N)(f)$ is a cycle of $C \otimes N$. So

$$f \longmapsto [(\sigma_{n-1} \otimes N)(f)] \in H_n(C \otimes N)$$ is the desired splitting.

$\square$