Lecture 2 (not abbreviated)

R-Modules, Resolutions, Hom-Ext, Tensor-Tor

To simplify things we only consider commutative rings R with a unit $1 \in R$.

2.1 Definition. R-module

This is an abelian group M together with a scalar multiplication

$$R \times M \longrightarrow M$$
$$(r, m) \longmapsto rm$$

s.t. all laws of an R-vectorspace hold; i.e.

$(r_1 + r_2)m = r_1 m + r_2 m$, $r(m_1 + m_2) = rm_1 + rm_2$,

$1 \cdot m = m$, $r_1(r_2 m) = (r_1 r_2)m$.

R-Homomorphism $f: M \longrightarrow N$ is a homom. of abelian group which is R-linear: $f(r \cdot m) = r \cdot (f(m))$

Whenever things get a little complicated we will assume that R is a principal ideal domain; i.e. R has no zero-divisors ($a \cdot b = 0 \Rightarrow (a = 0$ or $b = 0)$), and every ideal is principal ($I \subset R$ an ideal. Then $\exists r \in R$ s.t. $(r) = \{r \cdot s \mid s \in R\} = I$).

2.2 R PID (principal ideal domain), then submodules of free modules are free.

M is free iff $\exists$ $B \subset M$ s.t. every element of M can be uniquely written as a finite R-linear combination of elements of B. This is equivalent to:

If $F_B := \{ f : B \to R \mid f^{-1}(R \setminus \{0\}) \text{ is finite} \}$ with the obvious addition and scalar multiplication, then

$$f \longmapsto \sum_{b \in B} f(b) \cdot b \qquad \text{is an } R\text{-isomorphism.}$$

$F_B \to M$. $\square$

2.3 (Fundamental theorem of homological algebra)

2.3.a Let Res^f be the category with

objects free R-chain complexes F (i.e. chain complexes, such that all ∂_i are R-linear, and all F_i are free) with $F_i = 0$, $i < 0$, $H_i(F) = 0$, $i \neq 0$.

Morphisms: R-linear chain maps.

(The objects of Res^f are called free resolutions)

2.3.b Let $hRes^f$ be the category with the same objects as Res^f, but with morphisms the homotopy classes of chain maps (chain homotopies are required to be R-linear)

2.3.c Recall: $f_0 \underset{h}{\simeq} f_1 : F \to F'$, $g_0 \underset{h'}{\simeq} g_1 : F' \to F''$

then $g_0 \circ f_0 \underset{g_0 h + h' f_0}{\simeq} g_1 \circ f_1 : F \to F''$

Thus $[g] \circ [f] := [g \circ f]$ is well-defined where $[f]$ is the homotopy class of f.

2.3.d Let M be an R-module. A free resolution of M is a free resolution F together with a chain map $F \xrightarrow{p} M$ s.t.

$H(p) : H(F) \to H(M)$ is an isom.

(Any R-module M is an R-chain complex $\tilde{M}$ with $\tilde{M}_i = \begin{cases} M & i=0 \\ 0 & i \neq 0 \end{cases}$; we usually write M instead of $\tilde{M}$)

So a free resol. of M looks like :

$$\begin{tikzcd}
\cdots \arrow[r] & F_2 \arrow[r, "\partial_2"] \arrow[d, "p_2"] & F_1 \arrow[r, "\partial_1"] \arrow[d, "p_1"] & F_0 \arrow[r] \arrow[d, "p_0"] & 0 \\
& 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

p is a chain map iff $p_0 \circ \partial_1 = 0$

It induces an iso. in homology iff

$\ker p_0 = \operatorname{im} \partial_1$ and p_0 is an epimorphism.

2.3.e Since chain homotopic chain maps induce the same map in homology

$$\begin{aligned}
H_0 : hRes^f &\longrightarrow R\text{-Mod} \\
F &\longmapsto H_0(F) \\
F \xrightarrow{[a]} F' &\longmapsto H_0(a)
\end{aligned}$$

is a well defined functor

Fundamental Theorem.

$$H_0 : hRes^f \longrightarrow R\text{-Mod}$$

is an equivalence of categories. i.e.

(i) for any $M \in R\text{-Mod}$ $\exists$ $F \in hRes^f$

and an isomorphism $H_0(F) \cong M$

(i.e. $\exists$ a free resolution of M)

(ii) for any $F, F' \in R\text{-Mod}$

$$H_0 : \mathrm{Mor}(F, F') \longrightarrow \mathrm{Mor}(H_0(F), H_0(F'))$$

is bijective (in our case an isom. of R-modules)

(i) Start with $F_M =: F_0$, and $F_M \xrightarrow{p_0} M$

mapping the basis M of F_M identically to M.

Set $F_1 = F_{\ker p_0}$ and $\partial_1 : F_1 \longrightarrow F_0$

mapping elements of the basis to the corresponding element in $\ker p_0 \subset F_0$.

Assume $\underbrace{\ }_{i \geq 2 \text{ and}}$ $\partial_{i-1}, \partial_{i-2}, \ldots, \partial_1$ are defined

s.t. $H_k(F) = 0$ for $0 < k < i-1$

Set $F_i = F_{\ker \partial_{i-1}}$, $\partial_i : F_i \longrightarrow F_{i-1}$

maps $x \in \ker \partial_{i-1}$ considered as basis element of F_i to $x \in \ker \partial_{i-1} \subset F_{i-1}$.

Then $H_{i-1}(F) = 0$.

Since $H_0(p): H_0(F) \longrightarrow M$ is an isom., we are done

(ii) We have

$$\begin{tikzcd}
\cdots \arrow[r] & F_3 \arrow[r, "\partial_3"] & F_2 \arrow[r, "\partial_2"] & F_1 \arrow[r, "\partial_1"] & F_0 \arrow[r, "{[\ ]}"] & H_0(F) \arrow[r] \arrow[d, dashed, "\varphi"] & 0 \\
\arrow[r] & F_3' \arrow[r, "\partial_3'"] & F_2' \arrow[r, "\partial_2'"] & F_1' \arrow[r, "\partial_1'"] & F_0' \arrow[r, "{[\ ]}"] & H_0(F') \arrow[r] & 0
\end{tikzcd}$$

with exact rows.

(a) Given $\varphi: H_0(F) \longrightarrow H_0(F')$, define

$f_0: F_0 \to F_0'$ as follows. $B_0 \subset F_0$ basis

$b \in B_0$. $\exists$ $x' \in F_0'$ s.t. $[x'] = \varphi[b]$

set $p_0(b) := x'$: This defines p_0 s.t.

$\varphi \circ [\ ] = [\ ] \circ p_0$. Assume $i > 1$

and $p_{i-1}, \dots, p_0$ defined such that everything commutes that is visible.

Choose basis $B_i \subset F_i$, $b \in B_i$; then

$$\partial_{i-1}' p_{i-1} \partial_i (b) = p_{i-2} \partial_{i-1} \partial_i (b) = 0.$$

By exactness $\exists$ $x' \in F_i'$ s.t. $\partial_i'(x') = p_{i-1}\partial_i(b)$

Set $p_i(b) = x'$. Then $\partial_i' p_i(b) = p_{i-1}\partial_i(b)$

so $\partial_i' p_i = p_{i-1}\partial_i$.

(b) If $f, g: F \to F'$ are chain maps

z.z. $H_0(f)=H_0(g)$ define inductively a chain homotopy $h_k : F_k \longrightarrow F'_{k+1}$ starting with $h_0 : F_0 \longrightarrow F'_1$

$$\begin{tikzcd}
F_1 \arrow[r, "\partial_1"] \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "g_1"] & F_0 \arrow[r, "{[\ ]}"] \arrow[dl, "h_0"'] \arrow[d, shift right, "f_0"'] \arrow[d, shift left, "g_0"] & H_0(F) \arrow[d, "H_0(f)=H_0(g)"] \\
F_1' \arrow[r, "\partial_1'"] & F_0' \arrow[r, "{[\ ]}"] & H_0(F')
\end{tikzcd}$$

$b \in B_0$, $\quad [f_0(b)] = H_0(f)[b] = H_0(g)[b]$

$$= [g_0(b)]$$

$\Rightarrow \quad (g_0 - f_0)(b) \in \operatorname{im} \partial_1'$; i.e. $\exists\, x_b' \in F_1'$

with $\partial_1'(x_b') = (g_0 - f_0)(b)$. Set $h_0(b) = x_b'$.

Then $\partial_1' h_0(b) + \underbrace{h_{-1} \circ \partial_0(b)}_{=0} = (g_0 - f_0)(b)$

This holds for all basis elements ; thus

$$\partial_1' h_0 + h_{-1} \partial_0 = g_0 - f_0 .$$

Assume $i > 0$ $\quad h_{i-1}, \dots, h_0$ constructed s.t.

$$\partial_{k+1} h_k + h_{k-1} \partial_k = g_k - f_k , \quad 0 \le k < i .$$

Pick $b \in F_i$ $\quad$ Look at $\quad (g_i - f_i - h_{i-1} \partial_i)(b)$

This is a cycle in F'. Since $i > 0$ $\exists\, x_b' \in F_{i+1}'$

i.e. $\partial'_{i+1}(x'_b) = (g_i - f_i - h_{i-1}\partial_i)(b)$

Set $h_i(b) = x'_b$. Then

$\partial'_{i+1} h_i(b) + h_{i-1}\partial_i(b) = (g_i - f_i)(b)$. This holds for all basis elements and so

$$\partial'_{i+1} h_i + h_{i-1}\partial_i = g_i - f_i$$

and the inductive step is completed. $\square$

2.4 Hom - Ext. M, N R-modules

(F, p) a free resolution of M. Then

$F^*(N)$ is a cochain complex

$$F^i(N) = \mathrm{Hom}_R(F_i, N)$$

$$\delta^{i+1} : F^i(N) \longrightarrow F^{i+1}(N)$$
$$f \longmapsto (-1)^{i+1} f \circ \partial_{i+1}$$

Define $\mathrm{Ext}^i_R(M, N) := H^i(F^*(N))$

$(= H^i(F; N))$

(2.4.a) $M \xrightarrow{\alpha} M'$ R-homom.

F res. of M, F' of M', $[a] : F \to F'$ the unique homotopy class with

$$H_0(a) = \alpha \left(\text{i.e.} \begin{array}{ccc} F_0 & \xrightarrow{a_0} & F_0' \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p_0'} \\ M & \xrightarrow{\alpha} & M' \end{array} \quad \text{commutes} \right)$$

Then $a, a' \in [a]$ implies $\mathrm{Hom}(a, N)$ and $\mathrm{Hom}(a', N)$ are cochainhomotopic, and thus induce the same maps

$$\begin{array}{ccc} H^i(F'; N) & \longrightarrow & H^i(F; N) \\ \| & & \| \\ \mathrm{Ext}^i_R(M', N) & \longrightarrow & \mathrm{Ext}^i_R(M, N) \\ & \uparrow & \\ & \mathrm{Ext}(\alpha, N) := & \end{array}$$

In particular: F, F' free res. of M, then id_M induces unique <u>isom.</u> $H^i(F'; N) \to H^i(F; N)$

So, $\mathrm{Ext}^i(M, N)$ is well defined up to canonical isomorphism.

2.4.b $\qquad \mathrm{Ext}^0(M, N) \cong \mathrm{Hom}(M, N)$

$$\begin{array}{ccccccc} \leftarrow \mathrm{Hom}(F_1, N) & \xleftarrow{\delta^1} & \mathrm{Hom}(F_0, N) & \leftarrow & 0 \\ & & \uparrow {\scriptstyle \mathrm{Hom}(p_0, N)} & & \\ 0 & \longleftarrow & \mathrm{Hom}(M, N) & \leftarrow & 0 \end{array}$$

$f: M \to N$ induces $f \circ p_0 : F_0 \longrightarrow N$

Since $\operatorname{im}(\partial_1 : F_1 \to F_0) = \ker p_0$

$f \circ p_0 \in \ker \delta^1 = H^0(F; N)$. This map is obviously injective (since p_0 is surjective)

Conversely, let $f : F_0 \longrightarrow N$ be in $\ker \delta^1$. This means that $f|_{\operatorname{im}\partial_1} = 0$. Define

$\bar{f} : M \to N$ by mapping m to $f(p_0^{-1}(m))$.

Since $\ker p_0 = \operatorname{im}\partial_1$ this is well-defined; and

$\bar{f} \circ p_0 = f$ Thus $f \longmapsto f \circ p_0$ induces an isom.

$\operatorname{Hom}(M,N) \longrightarrow H_0(F; N)$. $\square$

If R is a PID every M has a short resolution given by

$$\begin{array}{ccccccc} 0 & \to & F_1 & \xrightarrow{\partial_1} & F_0 & \to & 0 \\ & & \downarrow & & \downarrow p_0 & & \\ & & 0 & \to & M & \to & 0 \end{array}$$

with p_0 surj., F_0 free, $F_1 = \ker p_0$, $\partial_1 : F_1 \hookrightarrow F_0$.

Thus $\operatorname{Ext}^i_R(M,N) = 0$ $i > 1$. Then, instead of Ext^1_R we simply write Ext

2.5 Tensor – Tor

2.5.a M free with basis $\mathcal{B}$. The pair will be denoted by $M_{\mathcal{B}}$. We define

$$M_{\mathcal{B}} \otimes N := \bigoplus_{b \in \mathcal{B}} N = \{ f: \mathcal{B} \to N \mid f^{-1}(N \setminus \{0\}) \text{ is finite} \}$$

with the obvious addition + scalar multipl.

If M, M' are free with bases $\mathcal{B}$ respectively $\mathcal{B}'$ and $\alpha: M \to M'$ an R-homom. we define

$$(\alpha \otimes N): M_{\mathcal{B}} \otimes N \longrightarrow M_{\mathcal{B}'} \otimes N \qquad \text{by}$$

$$(\alpha \otimes N)(f)(b') = \sum_{b \in \mathcal{B}} r_{b'b} \cdot f(b) \ , \ b' \in \mathcal{B}'$$

where $r_{b'b} \in R$ is defined (uniquely) by

$$\alpha(b) = \sum_{b' \in \mathcal{B}'} r_{b'b} \, b'$$

In particular if $M = M'$ and $\alpha = id_M$ there is a unique R-isomorphism

$$M_{\mathcal{B}} \otimes N \longrightarrow M_{\mathcal{B}'} \otimes N$$

In this sense $M_{\mathcal{B}} \otimes N$ is independent of the choice of basis; we may write $M \otimes N$ (which is a concrete module after a choice of basis of M)

Obviously: $\quad id_M \otimes N : M_B \otimes N \longrightarrow M_B \otimes N$

is the identity of $M_B \otimes N$ and if

$\beta : M' \longrightarrow M''$ is an R-homom then

$$(\beta \circ \alpha \otimes N) : M_B \otimes N \longrightarrow M''_{B''} \otimes N$$

is the composition

$$M_B \otimes N \xrightarrow{\alpha \otimes N} M'_{B'} \otimes N \xrightarrow{\beta \otimes N} M''_{B''} \otimes N$$

If F is a free R-chain complex, then

$F \otimes N$ is up to natural isomorphisms well defined by

$$(F \otimes N) = F_n \otimes N \qquad \text{and}$$

$$F_n \otimes N \longrightarrow F_{n-1} \otimes N \quad \text{is} \quad \partial_n \otimes N .$$

Clearly $F \otimes N$ is a chain complex; if

$f : F \to F'$ is a chain map, then $f \otimes N$ is a chain map, and if $f, g : F \to F'$ are chain homotopic then $f \otimes N$ and $g \otimes N$ are chain homotopic. Therefore, we may define (well defined up to canonical isomorphisms)

2.5.6 Let M, N be R-modules, we define

$\mathrm{Tor}_n^R(M,N)$ as follows: Let (F, p) be a free resolution of M, then

$$\mathrm{Tor}_n^R(M,N) = H_n(F \otimes N)$$

If $\alpha : M \to M'$ is an R-homomorphism then

$$\mathrm{Tor}_n^R(\alpha, N) : \mathrm{Tor}_n^R(M,N) \longrightarrow \mathrm{Tor}_n^R(M', N)$$

is $H_n(a \otimes N)$ for any chain map $a : F \to F'$ between free resolutions of M and M' inducing α.

If M is free, choosing the resolution $M \xrightarrow{id} M$ we see that $H_0(M,N) \cong M \otimes N$, i.e.

$\mathrm{Tor}_0^R(M,N) = M \otimes N$; therefore we define for any R-module M

Def. $M \otimes N := \mathrm{Tor}_0^R(M,N)$

If R is a PID we have $\mathrm{Tor}_i^R(M,N) = 0 \ \forall i > 1$.
The only remaining interesting module is $\mathrm{Tor}_1^R(M,N)$. We denote this by $\mathrm{Tor}^R(M,N)$ and call it the torsion product of M and N and write often simply $M * N$