This is an immediate corollary of :

Let $\varphi : A \longrightarrow A'$ be a homomorphism of ab gps

and (F, ρ) , (F', ρ') short free resolutions of A resp. A'.

Then there exists up to chain homotopy a unique

chain map $f : F \to F'$ such that

$$\begin{array}{ccc} F_0 & \xrightarrow{f_0} & F_0' \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho'} \\ A & \xrightarrow{\varphi} & A' \end{array}$$

commutes. In particular φ induces a unique

map $H(f) : H(F) \longrightarrow H(F')$ of homology groups

By definition p induces $H_0(F) \xrightarrow{\cong} A$

Consider $F^*(M)$. Then $H^0(F;M) \cong Hom(A,M)$
$\uparrow$ induced by p

and we define $Ext(A,M) := H^1(F;M)$, well defined up to natural isomorphism.

<u>Universal coefficient theorem for cohomology</u>

C free complex, M ab. gp. Then $\exists$ natural short exact sequences (for each n one)

$$0 \to Ext(H_{n-1}(C), M) \to H^n(C;M) \xrightarrow{ev} Hom(H_n(C), M) \to 0$$

$$[f] \mapsto ([z_n] \mapsto f(z_n))$$

$f \in \ker \delta^{n+1}$, $z_n \in \ker \partial_n$.

Cup-Product : Notation $\sigma : \Delta^k \to X$

$\{i_0 < \dots < i_\ell\} \subset \{0, 1, \dots, k\}$ Then

$\sigma\langle i_0, \dots, i_\ell\rangle : \Delta^\ell \longrightarrow X$ is the composition

$\Delta^\ell \xrightarrow{i} \Delta^k \xrightarrow{\sigma} X$ where $\Delta^\ell \to \Delta^k$ is the unique affine map mapping e_j to e_{i_j}, $j = 0, \dots, \ell$.

Then one defines for $(f,g) \in C^m(X;R) \times C^n(X;R)$

$f \cup g \in C^{m+n}(X;R)$

$$(f \cup g)(\sigma) = (-1)^{m \cdot n} f(\sigma\langle 0, \dots, m\rangle) \cdot g(\sigma\langle m, \dots, m+n\rangle)$$

where $\sigma : \Delta^{n+m} \to X$ is a sing. $(m+n)$-simplex.

$-\cup-$ is bilinear, associativ and we have

$$\delta(f \cup g) = \delta f \cup g + (-1)^m f \cup \delta g$$

This implies that $-\cup-$ induces a well defined product

$$H^m(X;R) \times H^n(X;R) \longrightarrow H^{m+n}(X;R)$$

Relativ version: $A, B \subset X$ s.t. A, B open in $A \cup B$

$-\cup-$ induces product

$$H^m(X,A;R) \times H^n(X,B;R) \longrightarrow H^{m+n}(X, A\cup B; R)$$

graded commutativ:

$$[f] \cup [g] = (-1)^{n \cdot m} [g] \cup [f]$$

in $H^{m+n}(X, A \cup B; R)$

Fact:

$$H^m(\mathbb{R}^{m+n}, (\mathbb{R}^m \setminus 0) \times \mathbb{R}^n) \times H^n(\mathbb{R}^{m+n}, \mathbb{R}^m \times (\mathbb{R}^n \setminus 0))$$

$$\Big\downarrow -\cup-$$

$$H^{m+n}(\mathbb{R}^{m+n}, \mathbb{R}^{m+n} \setminus \{0\})$$

maps (γ^m, γ^n) to γ^{m+n} where $\gamma^m, \gamma^n, \gamma^{m+n}$ are generators of the corresponding cohomology groups

This we used to compute the product structure of

$$H^*(\mathbb{R}P^n; \mathbb{Z}/2) \cong \mathbb{Z}/2[x]/\langle x^{n+1}\rangle, \quad H^*(\mathbb{C}P^n; \mathbb{Z}) \cong \mathbb{Z}[y]/\langle y^{n+1}\rangle$$

$x \in H^1(\mathbb{R}P^n; \mathbb{Z}/2)$ a generator $\quad$ $y \in H^2(\mathbb{C}P^n; \mathbb{Z})$ a generator.