Lecture 1.

Concise recap of contents of Topology 2 last semester

1. Simplicial complexes. We now like to see them as geometric realizations of abstract complexes.

<u>Abstract complex:</u> Set V, the set of vertices, and $\mathcal{K}$ a set of finite subsets of V s.t.

(i) $\sigma \in \mathcal{K}, \tau \subset \sigma \Rightarrow \tau \in \mathcal{K}$

(ii) $\forall v \in V \quad \{v\} \in \mathcal{K}$

We usually write $\mathcal{K}$ instead of $(V, \mathcal{K})$

$\mathcal{K}$ is <u>locally finite</u> if $\forall v \in V$ the set $\{\sigma : v \in \sigma\}$ is finite

$$|\mathcal{K}| = \{\lambda : V \longrightarrow [0,1] : \lambda^{-1}(0,1] \in \mathcal{K} \text{ and } \sum_v \lambda(v) = 1\}$$

Topology: $\sigma \in \mathcal{K}$, define $|\sigma| = \{\lambda \in \mathcal{K} : \lambda(v) = 0 \text{ if } v \notin \sigma\}$

Let $\sigma = \{v_0, \dots, v_n\}$ and Δ^n the standard n-simplex

Then $|\sigma| \longrightarrow \Delta^n$, $\lambda \longmapsto \sum_{i=0}^{n} \lambda(v_i) \cdot e_i$, is bijective.

Topologize $|\sigma|$ so that this map is a homeomorphism (This is independent of the ordering of the vertices of σ)

Then $A \subset |\mathcal{K}|$ is defined to be closed iff $A \cap |\sigma|$ is closed in $|\sigma|$ for all $\sigma \in \mathcal{K}$

$\sigma \in \mathcal{K}$, $|\mathring{\sigma}| = \{\lambda \in |\sigma| : \lambda(v) > 0 \ \forall v \in \sigma\}$, is called the interior of $|\sigma|$, $|\mathring{\sigma}|$ is also called an open simplex, $|\sigma|$ a simplex. If $\sigma = \{v_0, \dots, v_n\}$, then σ and $|\sigma|$ are called n-simplices of $\mathcal{K}$ resp. $|\mathcal{K}|$.

$$\mathring{st}(\sigma, \mathcal{K}) := \bigcup_{\sigma \subset \tau} |\mathring{\tau}| \subset |\mathcal{K}| \qquad \text{: open star of } \sigma \text{ in } |\mathcal{K}|$$

$$st(\sigma, \mathcal{K}) := \{\tau \in \mathcal{K} : \sigma \subset \tau\} \subset \mathcal{K} \text{ : star of } \sigma \text{ in } \mathcal{K}$$

$$lk(\sigma, \mathcal{K}) := \{\tau \in st(\sigma, \mathcal{K}) : \sigma \cap \tau = \emptyset\}$$

Facts: $\mathring{st}(\sigma, \mathcal{K})$ is an open nbhd. of $|\mathring{\sigma}|$ in $|\mathcal{K}|$

$|st(\sigma, \mathcal{K})|$ and $\mathring{st}(\sigma, \mathcal{K})$ are contractible

$$|st(\sigma, \mathcal{K})| \cong |\sigma| \underset{\uparrow \text{join}}{*} |lk(\sigma, \mathcal{K})|$$

A, B top. spaces $\quad A * B = A \times I \times B / \sim$

with $\sim$ generated by
$$(a, 0, b) \sim (a', 0, b) \qquad a', a \in A,\ b \in B$$
$$(a, 1, b) \sim (a, 1, b') \qquad a \in A,\ b, b' \in B$$

we often write σ for $|\sigma|$. we always write v for $|\{v\}|$.

Simplicial maps: $\mathcal{K}, \mathcal{K}'$ abstract complexes. A simpl. map $f : \mathcal{K} \to \mathcal{K}'$ is a map $f : V \longrightarrow V'$ of the underlying vertex sets s.t. $\forall \sigma \in \mathcal{K}$ $f(\sigma) \in \mathcal{K}'$; f need not be injective. The geom. real. of f is

$$|f| : |\mathcal{K}| \longrightarrow |\mathcal{K}'| \quad \text{given by}$$

$$|f|(\lambda) : V' \longrightarrow [0,1]$$
$$v' \longmapsto \sum_{v \in f^{-1}(v')} \lambda(v),$$

for $\lambda : V \to [0,1]$ an element of $|\mathcal{K}|$

$|f|$ is continuous. Any map $|\mathcal{K}| \to |\mathcal{K}'|$ of this type is also called a simplicial map.

2. Δ-complexes. Δ-structure on top. space X is family of continuous maps

$$f_\alpha : \Delta^{n(\alpha)} \longrightarrow X \quad , \quad \alpha \in A, \quad \text{s.t.}$$

(i) $f_\alpha | \mathring{\Delta}^{n(\alpha)}$ is injective, and

$$f_\alpha(\mathring{\Delta}^{n(\alpha)}) \cap f_\beta(\mathring{\Delta}^{n(\beta)}) = \emptyset \quad \text{for } \alpha \neq \beta$$

(ii) $X = \bigcup_{\alpha \in A} f_\alpha(\mathring{\Delta}^{n(\alpha)})$

(iii) If $\varepsilon_i^{n-1} : \Delta^{n-1} \longrightarrow \Delta^n$, $0 \le i \le n$, is given by

$$\varepsilon_i^{n-1}(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$$

then the following holds: for every α and $0 \le i \le n(\alpha)$ $f_\alpha \circ \varepsilon_i^{n(\alpha)-1}$ belongs to the Δ-structure.

(iv) $A \subset X$ is closed iff $f_\alpha^{-1}(A)$ is closed in $\Delta^{n(\alpha)}$.

Example: $(V, \mathcal{K})$ abstract complex, „$\le$" partial order on V s.t. every $\sigma \in \mathcal{K}$ is totally ordered. One gets a Δ-structure on $|\mathcal{K}|$ as follows. Let $\sigma \in \mathcal{K}$, $\sigma = \{v_0 < v_1 < \dots < v_n\}$

Define $f_\sigma : \Delta^n \longrightarrow |\sigma| \subset |\mathcal{K}|$

by $f_\sigma(t_0, \dots, t_n)(v_i) = t_i$, $0 \le i \le n$.

$f_\sigma : \Delta^n \longrightarrow |\sigma|$ is a homeomorphism, $f_\sigma : \Delta^n \longrightarrow |\mathcal{K}|$ a closed embedding.

3. Chain complexes   a sequence

$$\cdots \xrightarrow{\partial_{i+1}} A_i \xrightarrow{\partial_i} A_{i-1} \xrightarrow{\partial_{i-1}} \cdots \qquad i \in \mathbb{Z}$$

of homomorphisms $\partial_i : A_i \to A_{i-1}$ of ab. gps with

$$\partial_{i-1} \circ \partial_i = 0 \quad \text{for all } i$$

Notation $A = (A_i, \partial_i)$

$Z_i(A) := \ker \partial_i$ i-th cycle group

$B_i(A) := \operatorname{im} \partial_{i+1}$ i-th boundary group

∂_i i-th boundary map

$$H_i(A) := Z_i(A) / B_i(A)$$ i-th homology gp of A

(A_i, ∂_i^A), (B, ∂_i^B) chain complexes. Chain map

$f : A \to B$ is family of homom. $f_i : A_i \to B_i$

s.t.

$$\begin{array}{ccc} A_i & \xrightarrow{\partial_i^A} & A_{i-1} \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f_{i-1}} \\ B_i & \xrightarrow{\partial_i^B} & B_{i-1} \end{array}$$

commutes for all i.

f chain map. Then $f_i(Z_i(A)) \subset Z_i(B)$

$$f_i(B_i(A)) \subset B_i(B)$$

Thus f induces for all i homom.

$$H_i(f) : H_i(A) \to H_i(B)$$
$$(z + B_i(A)) \longmapsto (f_i(z) + B_i(B)), \; z \in Z_i(A).$$

Chain homotopy h between chain maps $f, g : A \to B$

is a family of homom. $h_i : A_i \to B_{i+1}$ s.t.

$$\partial_{i+1}^B h_i + h_{i-1} \partial_i^A = g_i - f_i \quad \text{for all } i$$

If f and g are chain homotopic then $H_i(f) = H_i(g)$ $\forall i$.

- If A is a free chain complex (i.e. all A_i are free abelian) and homomorphisms $\varphi_i : H_i(A) \to H_i(B)$ are given then there exists up to chain homotopy a unique chain map $f : A \to B$ s.t. $H_i(f) = \varphi_i$.

- A chain map $f : A \to B$ between free complexes which induces isomorphisms $H_i(f)$ $\forall i$ is a chain homotopy equivalence

Singular homology:

X top. space. $C_i(X)$ = free abelian group over the set of singular i-simplices, i.e. over $\{\sigma : \Delta^i \to X : \sigma \text{ continuous}\}$

$\partial_i : C_i(X) \to C_{i-1}(X)$ defined on basis elements $\sigma : \Delta^i \to X$

by $\partial_i \sigma = \sum_{j=0}^{i} (-1)^j \sigma \circ \varepsilon_j^{i-1}$, $\varepsilon_j^{i-1} : \Delta^{i-1} \to \Delta^i$ as defined on page 1.3

$C(X) = (C_i(X), \partial_i)$ is a chain complex (singular complex of X)

and $H_i(X) := H_i(C(X))$; i-th sing. homology group of X.

- Cone construction. If $CX := X \times I / X \times \{1\}$ is the cone over X then CX is chain contractible. i.e. the inclusion of $\mathbb{Z} \xrightarrow{i} CX$ is a chain homotopy equivalence. Here, any abelian gp. A is considered as
- a chain complex, where all chain groups are 0 except the 0-th chaingroup which is A.

  i maps $1 \in \mathbb{Z}$ to $\sigma^0 : \Delta^0 \to$ cone point $[X \times \{1\}] \in CX$

If $CX \xrightarrow{p} \mathbb{Z}$ is the obvious map which maps all sing. 0-simplices to $1 \in \mathbb{Z}$

one needs a chain homotopy between $i \circ p$ and id_{CX}. This is given on basis elements $\sigma : \Delta^n \longrightarrow CX$ by

$h_n \sigma : \Delta^{n+1} \longrightarrow CX$

$$h_n \sigma (t_0, \dots, t_{n+1}) = \begin{cases} \left[\sigma\left(\frac{t_1}{1-t_0}, \dots, \frac{t_{n+1}}{1-t_0}\right), t_0\right], & 0 \le t_0 < 1 \\ [X \times \{1\}] & t_0 = 1 \end{cases}$$

- acyclic model constructions.

(i) $f, g : X \to Y$ homotopic, then $C(f), C(g) : C(X) \to C(Y)$ chain homotopic

(ii) barycentric subdivision operator $S : C(X) \to C(X)$

Each construction is done inductively on $C_n(X)$. For each n one starts by describing the construction on the singular n-simplex $\iota_n : \Delta^n \xrightarrow{id} \Delta^n$ of Δ^n and then extends functorially to all of $C_n(X)$ for all X.

Example (i) We want to prove that the 2 maps

$$id_X^0, id_X^1 : X \to X \times I \quad , \quad id_X^i(x) = (x, i),$$

induce chain homotopic maps $C(id_X^0), C(id_X^1)$.

We start with $n = 0$, $X = \Delta^0$,

$$C_0(\Delta^0) \xrightarrow{h_0(\Delta^0)} C_1(\Delta^0 \times I)$$

$$\iota_0 \longmapsto \left(h_0(\Delta^0)\iota_0 : \Delta^1 \to \Delta^0 \times I\right)$$

$$(t_0, t_1) \longmapsto (1, t_1)$$

Then

$$\partial_1 h_0(\Delta^0)(\iota_0) = C(\iota_0^1)(\iota_0) - C(\iota_0^0)(\iota_0)$$

Since $\partial_0(\iota_0) = 0$ we have

$$\partial_1 h_0(\Delta^0) + h_{-1}(\Delta^0)\partial_0 = C(\iota_0^1) - C(\iota_0^0)$$

where for all spaces X we define $h_i(X) : C_i(X) \to C_{i+1}(X \times I)$ to be 0 for $i < 0$.

Now we define $h_0(X) : C_0(X) \to C_0(X \times I)$ for all X by: given $\sigma : \Delta^0 \longrightarrow X$ then

$$h_0(X)(\sigma) = C_1(\sigma \times id_I)(h_0(\Delta^0))$$

and have

$$\partial_1 h_0(X)(\sigma) = C_0(\sigma \times id_I)(\partial_1 h_0(\Delta^0))$$

$$= C_0(\sigma \times id_I)(\iota_0^1 - \iota_0^0)$$

$$= C_0(id_X^1)(\sigma) - C_0(id_X^0)(\sigma)$$

One checks: if $f : X \to Y$ is continuous then

$$h_0(Y) \circ C_0(f) = C_1(f) \circ h_0(X).$$

Assume that $h_k(X) : C_k(X) \longrightarrow C_{k+1}(X \times I)$ has been defined for $k < n$ such that

$\partial_{k+1} h_k(X) + h_{k-1}(X)\partial_k = C_k(id_X^1) - C_k(id_X^0)$ and

(*) $h_k(Y) \circ C_k(f) = C_{k+1}(f) \circ h_k(X)$ for $f : X \to Y$

we want to define $\forall X \quad h_n(X)$ s.t. $(*)$ holds.

We first look at $C_n(\Delta^n)$, in particular at

$\iota_n \in C_n(\Delta^n)$. We want to define $h_n(\Delta^n)(\iota_n)$

s.t.

$$\partial_{n+1} h_n(\Delta^n)(\iota_n) + h_{n-1}(\Delta^n)(\partial_n \iota_n) = C_n(\iota_n^1)(\iota_n) - C_n(\iota_n^0)(\iota_n)$$
$$(= \iota_n^1 - \iota_n^0) \in C_n(\Delta^n \times I)$$

Now we use that $H_n(\Delta^n \times I) = 0$ (acyclicity)

We need to know whether

$d_n := C_n(\iota_n^1)(\iota_n) - C_n(\iota_n^0)(\iota_n) - h_{n-1}(\Delta^n)(\partial_n \iota_n)$ is a cycle.

Then it is a boundary of an $(n+1)$-chain of $\Delta^n \times I$.

Choose one, and call it $h_n(\Delta^n)(\iota_n)$.

$$\partial_n d_n = C_{n-1}(\iota_n^1)(\partial \iota_n) - C_{n-1}(\iota_n^0)(\partial \iota_n)$$
$$- \partial_n h_{n-1}(\Delta^n)(\partial_n \iota_n)$$
$$\overset{(*)}{=} \partial_n h_{n-1}(\Delta^n)(\partial \iota_n) + h_{n-2}(\Delta^n)\partial_{n-1}(\partial \iota_n)$$
$$- \partial_n h_{n-1}(\Delta^n)(\partial \iota_n) = 0.$$

Therefore we find $h_n(\Delta^n)(\iota_n)$ as desired. We define now

$$h_n(X) : C_n(X) \longrightarrow C_{n+1}(X) \quad \text{by}$$

$$h_n(X)(\sigma) = C_{n+1}(\sigma \times id_I)\big(h_n(\Delta^n)(\iota_n)\big) \quad \text{for}$$

$\sigma : \Delta^n \to X$.

and one easily checks that

(*) holds for $k = n$.

Homotopy invariance: $f_0, f_1 : X \to Y$ homotopic,

$F : X \times I \to Y$ homotopy (i.e. $F \circ id_X^i = f_i$, $i = 0,1$)

Then

$C_{n+1}(F) \circ h_n(X) : C_n(X) \longrightarrow C_{n+1}(Y)$ is a

chain htpy from $C(f_0) = C(F \circ id_X^0) = C(F) \circ C(id_X^0)$

$C(f_1) = C(F \circ id_X^1)$. □

All we needed is: (i) $C_n(X)$ is free abelian with

basis all continuous maps from Δ^n to X

(ii) $H_n(\Delta^n \times I) = 0$ for $n > 0$

The situation above can be described also as follows.

We have two functors from TOP to Chain

$F \quad X \longmapsto C(X)$

$G \quad X \longmapsto C(X \times I)$

And we have two natural transformations $F \longrightarrow G$

α^0, α^1 with $C(X) \xrightarrow{\alpha^i} C(X \times I)$

given by $\sigma \longmapsto C(id_X^i)(\sigma)$

α^0 and α^1 induce the same map in H_0
since for any singular 0-simplex $\sigma : \Delta^0 \to X$
the simplices $id_X^0 \circ \sigma$ and $id_X^1 \circ \sigma$ are homologous
(Their images are the endpoints of the path
$t \longmapsto (\sigma(\Delta^0), t)$).

Then, if for each $k \geq 0$ we have a set $\mathcal{M}_k$ of Top. spaces and a set $A_k = \{ a_k \in F_k(M) \}$ with $M \in \mathcal{M}_k$
s.t. $F_k(X)$ is free abelian with basis

$$\{ F_k(f)(a_k) \ : \ f : M \to X \ , \ a_k \in A_k \cap F_k(M) \}$$

and for $k > 0$ $\quad H_k(G(M)) = 0 \quad M \in \mathcal{M}_k$
Then α^0 and α^1 are naturally chain homotopic.

Anyway: (i) on page 1.6 gives htpy invariance
(ii) gives the important excision property of singular homology

(X,A) pair in Top ; $\quad C(X,A) = (C_n(X,A), \partial_n)_{n\in\mathbb{Z}}$

$C_n(X,A) := C_n(X) / C_n(A)$, ∂_n the induced boundary map. Then we have:

Excision: If $U \subset A$ s.t. $\overline{U} \subset \operatorname{Int} A$, then the inclusion

$(X \setminus U, A \setminus U) \longrightarrow (X, A)$ induces isomorphisms for all homology groups.

(i) This we used to prove: Let X be a Δ-complex; then the maps of the Δ-structure define singular simplices of X. The subgroup of $C_n(X)$ generated by the n-simplices of the Δ-structure is denoted by $\Delta_n(X)$.

$\partial_n(\Delta_n(X)) \subset \Delta_{n-1}(X)$ because of condition (iii) on page 1.3. The corresponding homology groups $H_n^\Delta(X)$ are called the simplicial homology groups of X.

<u>Claim:</u> $\Delta(X) \hookrightarrow C(X)$ is a chain homotopy equivalence.

~~and~~

(ii) Mayer-Vietoris $X = X_1 \cup X_2$ with X_1, X_2 open

Then $\exists$ natural exact sequence

$$\cdots \longrightarrow H_{i+1}(X) \xrightarrow{\partial_{i+1}} H_i(X_1 \cap X_2) \xrightarrow{\binom{j_1}{-j_2}} H_i(X_1) \oplus H_i(X_2) \longrightarrow$$

$$\longrightarrow H_i(X) \xrightarrow{\partial_i} \cdots$$

$j_k : H_i(X_1 \cap X_2) \to H_i(X_k)$, $k = 1, 2$, induced by inclusion. This is a particular case of excision +

short exact sequence of chain complexes $\rightsquigarrow$

long exact sequence of homology groups

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \qquad \text{s.e.s. in chain}$$

$\exists \forall n \quad \partial_n : H_n(C) \longrightarrow H_{n-1}(A)$ s.t.

$$\cdots \to H_{n+1}(C) \xrightarrow{\partial_{n+1}} H_n(A) \xrightarrow{H_n(\alpha)} H_n(B) \xrightarrow{H_n(\beta)} H_n(C) \xrightarrow{\partial_n} \cdots$$

is exact.

<u>Cohomology with coeff. in M</u>

(C_n, ∂_n) chain complex ;

$$\begin{array}{ccc} C^n(M) := \mathrm{Hom}(C_n, M) & & f \\ \delta^{n+1} \big\downarrow & & \big\downarrow \\ & & (-1)^{n+1} f \circ \partial_{n+1} \\ C^{n+1}(M) := \mathrm{Hom}(C_{n+1}, M) & & \end{array}$$

$$H^n(C; M) := \ker \delta^{n+1} / \operatorname{im} \delta^n$$

<u>Ext (A, M)</u> , A, M abelian gps

$0 \to F_1 \xrightarrow{\partial_1} F_0 \to 0$ free chain complex

together with $F_0 \xrightarrow{p} A$ s.t. $0 \to F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{p} A \to 0$ is exact. Then (F, p) is called a short free resolution of A.

Up to natural homotopy equivalence each ab. gp has a unique short free resolution